

# MULTIFRACTAL ANALYSIS OF SUPERPROCESSES WITH STABLE BRANCHING IN DIMENSION ONE.

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ABSTRACT. We show that density functions of a  $(\alpha, 1, \beta)$ -superprocesses are almost sure multifractal for  $\alpha > \beta + 1$ ,  $\beta \in (0, 1)$  and calculate the corresponding spectrum of singularities.

## 1. INTRODUCTION, MAIN RESULTS AND DISCUSSION

For  $0 < \alpha \leq 2$  and  $1 + \beta \in (1, 2]$ , the so-called  $(\alpha, d, \beta)$ -superprocess  $X = \{X_t : t \geq 0\}$  in  $\mathbb{R}^d$  is a finite measure-valued process related to the log-Laplace equation

$$\frac{d}{dt}u = \Delta_\alpha u + au - bu^{1+\beta}, \quad (1.1)$$

where  $a \in \mathbb{R}$  and  $b > 0$  are any fixed constants. Its underlying motion is described by the fractional Laplacian  $\Delta_\alpha := -(-\Delta)^{\alpha/2}$  determining a symmetric  $\alpha$ -stable motion in  $\mathbb{R}^d$  of index  $\alpha \in (0, 2]$  (Brownian motion corresponds to  $\alpha = 2$ ), whereas its continuous-state branching mechanism

$$v \mapsto -av + bv^{1+\beta}, \quad v \geq 0, \quad (1.2)$$

belongs to the domain of attraction of a stable law of index  $1 + \beta \in (1, 2)$  (the branching mechanism is critical if  $a = 0$ ).

Let  $d < \frac{\alpha}{\beta}$ . Then, for any fixed time  $t$ ,  $X_t(dx)$  is a.s. absolutely continuous with respect to the Lebesgue measure (cf. Fleischmann [3] for  $a = 0$ ). Moreover, as it is shown in Fleischmann, Mytnik, and Wachtel [4, Theorem 1.2(a),(c)], there is a *dichotomy* for the density function of the measure (in what follows, we just say the “density of the measure”): There is a continuous version of the density of  $X_t(dx)$  if  $d = 1$  and  $\alpha > 1 + \beta$ , but otherwise the density is unbounded on open sets of positive  $X_t(dx)$ -measure. Note that the case of  $\alpha = 2$  had been studied earlier in Mytnik and Perkins [13]. In the case of continuity, Hölder regularity properties of the density had been studied in [4], [5].

From now on, we always assume that  $d = 1$ ,  $\alpha > 1 + \beta$ , that is, there is a continuous version of the density at fixed time  $t$ . This density, with a slight abuse of notation, will be also denoted by  $X_t(x)$ ,  $x \in \mathbb{R}$ .

Let us first recall the notion of pointwise regularity (see e.g. Jaffard [7]). We say that a function  $f$  has regularity of index  $\eta > 0$  at a point  $x \in \mathbb{R}$ , if there exists

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an open neighborhood  $U(x)$  of  $x$ , a constant  $C > 0$  and a polynomial  $P_x$  of degree at most  $\lfloor \eta \rfloor$  such that

$$|f(y) - P_x(y)| \leq C |y - x|^\eta \quad \text{for all } y \in U(x). \quad (1.3)$$

For  $\eta \in (0, 1)$  the above definition coincides with the definition of Hölder continuity with index  $\eta$  at a point. Note that sometimes the class of functions satisfying (1.3) is denoted by  $C^\eta(x)$ . Now, given  $f$  one would like to find the supremum over all  $\eta$  such that (1.3) holds for some constant  $C$  and polynomial  $P_x$ . This leads to the definition of so-called *optimal* Hölder exponent (or index) of  $f$  at  $x$ :

$$H_f(x) := \sup\{\eta > 0 : f \in C^\eta(x)\}, \quad (1.4)$$

and we set it to 0 if  $f \notin C^\eta(x)$  for all  $\eta > 0$ . To simplify the exposition we will sometimes call  $H_f(x)$  the Hölder exponent of  $f$  at  $x$ .

Let us fix  $t > 0$  and return to the continuous density  $X_t$  of the  $(\alpha, 1, \beta)$ -superprocess. In what follows,  $H_X(x)$  will denote the optimal Hölder exponent of  $X_t$  at  $x \in \mathbb{R}$ . In Theorem 1.2(a),(b) of [4], the so-called *optimal index* for *local* Hölder continuity of  $X_t$  had been determined by

$$\eta_c := \frac{\alpha}{1 + \beta} - 1 \in (0, 1). \quad (1.5)$$

This means that  $\inf_{x \in K} H_X(x) \geq \eta_c$  for any compact  $K$  and, moreover, in any non-empty open set  $U \subset \mathbb{R}$  with  $X_t(U) > 0$  one can find (random) points  $x$  such that  $H_X(x) = \eta_c$ . Moreover, it was proved in [5] that for any fixed point  $x \in \mathbb{R}$ , such that  $X_t(x) > 0$ , we have

$$H_X(x) = \bar{\eta}_c := \frac{1 + \alpha}{1 + \beta} - 1,$$

in the case of  $\beta > (\alpha - 1)/2$ , and  $H_X(x) \geq 1$  if  $\beta \leq (\alpha - 1)/2$ .

**Remark 1.1.** In [5] the classical definition of Hölder exponent was used, which can take only values between 0 and 1. Hence the result in [5] states that the optimal index of Hölder continuity (in classical sense) equals to  $\min\left\{\frac{1+\alpha}{1+\beta} - 1, 1\right\}$ , for any  $\beta \in (0, \alpha - 1)$ .

The purpose of this paper includes proving that on any open set of positive  $X_t$  measure, and for any  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$  there are, with probability one, (random) points  $x \in \mathbb{R}$  such that the optimal Hölder index  $H_X(x)$  of  $X_t$  at  $x$  is exactly  $\eta$ . Moreover, for an open set  $U \subset \mathbb{R}$ , we are going to establish the *Hausdorff dimension*, say  $D_U(\eta)$ , of the (random) set

$$\mathcal{E}_{U,X,\eta} \equiv \{x \in U : H_X(x) = \eta\}.$$

We will show that the function  $\eta \mapsto D_U(\eta)$  is independent of  $U$ ; it reveals the so-called *multifractal spectrum* related to the optimal Hölder index at points.

To formulate our main result we need also the following notation. Let  $\mathcal{M}_f$  denote the set of finite measures on  $\mathbb{R}$ , and for  $\mu \in \mathcal{M}_f$ ,  $|\mu|$  will denote the total mass  $\mu(\mathbb{R})$ . Our main result is as follows.

**Theorem 1.2 (Multifractal spectrum).** *Fix  $t > 0$ , and  $X_0 = \mu \in \mathcal{M}_f$ . Let  $d = 1$  and  $\alpha > 1 + \beta$ . Then, with probability one, for any open set  $U \subset \mathbb{R}$ ,*

$$D_U(\eta) = (\beta + 1)(\eta - \eta_c), \quad \eta \in [\eta_c, \bar{\eta}_c] \setminus \{1\},$$

*whenever  $X_t(U) > 0$ .*

**Remark 1.3.** We conjecture that the  $D_U(\bar{\eta}_c) = 1$ , and whenever  $\bar{\eta}_c \geq 1$ , the function  $D_U(\cdot)$  can be also continuously extended to  $\eta = 1$ , i.e.,  $D_U(1) = (\beta + 1)(1 - \eta_c)$ . We believe that  $D_U(\bar{\eta}_c) = 1$  can be proved using the same strategy as in the proof of Theorem 1.2, but some technical steps become much more difficult. If  $\bar{\eta}_c < 1$  then we can show it using the results from [5]. Indeed, for  $B$  being arbitrary ball in  $(0, 1)$  define

$$\lambda(\mathcal{E}_{B,X,\bar{\eta}_c}) = \int_B 1_{\{H_X(x)=\eta_c\}} dx$$

where  $\lambda$  is the Lebesgue measure on  $\mathbb{R}$ . Then, by the Fubini theorem, we have

$$\begin{aligned} \mathbf{E} \left[ \lambda(\mathcal{E}_{B,X,\bar{\eta}_c}) \mid \inf_{y \in B} X_t(y) > 0 \right] &= \mathbf{E} \left[ \int_B 1_{\{H_X(x)=\eta_c\}} dx \mid \inf_{y \in B} X_t(y) > 0 \right] \\ &= \int_B \mathbf{P}(H_X(x) = \eta_c \mid \inf_{y \in B} X_t(y) > 0) dx \\ &= \lambda(B), \end{aligned}$$

in the last step we used Theorem 1.1 from [5], which states that  $H_X(x) = \eta_c$  for every fixed point  $x$ . That is, given  $\{\inf_{y \in B} X_t(y) > 0\}$ , we get that, with probability one,  $D_B(\bar{\eta}_c) = 1$ . We may fix  $\omega$  outside a  $\mathbf{P}$ -null set so that this holds for any rational ball  $B$ , that is, for any ball with a rational radius and center. Let  $U$  be an arbitrary open set such that  $\{\inf_{y \in U} X_t(y) > 0\}$ . Then, there is always a rational ball  $B$  in  $U$  such that  $\{\inf_{y \in B} X_t(y) > 0\}$ , and so the result follows immediately from what we derived for the fixed ball.

**Remark 1.4.** The fact that our proof fails in the case  $\eta = 1$  is even more disappointing. Formally, it happens for some technical reasons, but one has also to note, that this point is critical: it is the borderline between differentiable and non-differentiable functions.

**Remark 1.5.** The condition  $\alpha > 1 + \beta$  excludes the case of the quadratic super-Brownian motion, i.e.,  $\alpha = 2$ ,  $\beta = 1$ . But it is known "folklore" result that the super-Brownian motion is almost surely monofractal, that is,  $\eta_c = \bar{\eta}_c = 1/2$ .

The multifractal spectrum of random functions and measures has attracted attention for many years and has been studied for example in Dembo et al. [1], Durand [2], Hu and Taylor [6], Klenke and Mörters [11], Le Gall and Perkins [12], Mörters and Shieh [15] and Perkins and Taylor [16]. The multifractal spectrum of singularities that describes the Hausdorff dimension of sets of different Hölder exponents of functions was investigated for deterministic and random functions in Jaffard [7, 8, 9] and Jaffard and Meyer [10].

After some preparation in the next section, the proof of Theorem 1.2 will be given in Sections 3, 4, 5.

## 2. PRELIMINARIES

In this section we collect some technical results from [4, 5].

Let  $p^\alpha$  denote the continuous  $\alpha$ -stable transition kernel related to the fractional Laplacian  $\Delta_\alpha = -(-\Delta)^{\alpha/2}$  in  $\mathbb{R}$ , and  $S^\alpha$  the related semigroup. Fix  $X_0 = \mu \in \mathcal{M}_f \setminus \{0\}$ .

First we want to recall the *martingale decomposition* of the  $(\alpha, 1, \beta)$ -superprocess  $X$  (valid for any  $\alpha \in (0, 2]$ ,  $\beta \in (0, 1)$ ; see, e.g., [4, Lemma 1.6]): For all sufficiently

smooth bounded non-negative functions  $\varphi$  on  $\mathbb{R}$  and  $t \geq 0$ ,

$$\langle X_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t ds \langle X_s, \Delta_\alpha \varphi \rangle + M_t(\varphi) + a I_t(\varphi) \quad (2.1)$$

with discontinuous martingale

$$t \mapsto M_t(\varphi) := \int_{(0,t] \times \mathbb{R} \times \mathbb{R}_+} \tilde{\mathcal{N}}(d(s, x, r)) r \varphi(x) \quad (2.2)$$

and increasing process

$$t \mapsto I_t(\varphi) := \int_0^t ds \langle X_s, \varphi \rangle. \quad (2.3)$$

Here  $\tilde{\mathcal{N}} := \mathcal{N} - \hat{\mathcal{N}}$ , where  $\mathcal{N}(d(s, x, r))$  is a random measure on  $(0, \infty) \times \mathbb{R} \times (0, \infty)$  describing all the jumps  $r\delta_x$  of  $X$  at times  $s$  at sites  $x$  of size  $r$  (which are the only discontinuities of the process  $X$ ). Moreover,

$$\hat{\mathcal{N}}(d(s, x, r)) = \varrho ds X_s(dx) r^{-2-\beta} dr \quad (2.4)$$

is the compensator of  $\mathcal{N}$ , where  $\varrho := b(1+\beta)\beta/\Gamma(1-\beta)$  with  $\Gamma$  denoting the Gamma function.

Under our assumptions the random measure  $X_t(dx)$  is a.s. absolutely continuous for every fixed  $t > 0$ . From the Green function representation related to (2.1) (see, e.g., [4, (1.9)]) we obtain the following representation of a version of the density function of  $X_t(dx)$  (see, e.g., [4, (1.12)]):

$$\begin{aligned} X_t(x) &= \mu * p_t^\alpha(x) + \int_{(0,t] \times \mathbb{R}} M(d(s, y)) p_{t-s}^\alpha(y-x) \\ &+ a \int_{(0,t] \times \mathbb{R}} I(d(s, y)) p_{t-s}^\alpha(y-x) =: Z^1(x) + Z^2(x) + Z^3(x), \quad x \in \mathbb{R}, \end{aligned} \quad (2.5)$$

(with notation in the obvious correspondence). Note that although  $Z^i, i = 1, 2, 3$ , depend on  $t$ , we omit the corresponding subscript since  $t$  is fixed throughout the paper.  $M(d(s, y))$  in (2.5) is the martingale measure related to (2.2) and  $I(d(s, y))$  the random measure related to (2.3). Note that by Lemma 1.7 of [4] the class of "legitimate" integrands with respect to the martingale measure  $M(d(s, y))$  includes the set of functions  $\psi$  such that for some  $p \in (1+\beta, 2)$ ,

$$\int_0^T ds \int_{\mathbb{R}} dx S_s^\alpha \mu(x) |\psi(s, x)|^p < \infty, \quad \forall T > 0. \quad (2.6)$$

We let  $\mathcal{L}_{\text{loc}}^p$  denote the space of equivalence classes of measurable functions satisfying (2.6). For  $\alpha > 1+\beta$ , it is easy to check that, for any  $t > 0, z \in \mathbb{R}$ ,  $(s, x) \mapsto p_{t-s}^\alpha(z-x) \mathbf{1}_{s < t}$  is in  $\mathcal{L}_{\text{loc}}^p$  for any  $p \in (1+\beta, 2)$ , and hence the stochastic integral in the representation (2.5) is well defined.

Throughout the paper we will need the following estimate for the  $\alpha$ -stable transition kernel  $p^\alpha$  (see [4, Lemma 2.1]).

**Lemma 2.1.** *For every  $\delta \in [0, 1]$ , there exists a constant  $C > 0$  such that*

$$|p_t^\alpha(x) - p_t^\alpha(y)| \leq C \frac{|x-y|^\delta}{t^{\delta/\alpha}} (p_t^\alpha(x/2) + p_t^\alpha(y/2)), \quad t > 0, \quad x, y \in \mathbb{R}. \quad (2.7)$$

By the methods very similar to those used for the proof of the previous lemma, one can get the following result.

**Lemma 2.2.** (a) For every  $\delta \in [0, 1]$ , there exists a constant  $C > 0$  such that

$$\left| \frac{\partial p_t^\alpha(x)}{\partial x} - \frac{\partial p_t^\alpha(y)}{\partial y} \right| \leq C \frac{|x-y|^\delta}{t^{(1+\delta)/\alpha}} (p_t^\alpha(x/2) + p_t^\alpha(y/2)), \quad t > 0, \quad x, y \in \mathbb{R}. \quad (2.8)$$

(b) There exists a constant  $C > 0$  such that

$$\left| \frac{\partial p_t^\alpha(x)}{\partial x} \right| \leq C t^{-1/\alpha} p_t^\alpha(x/2), \quad t > 0, \quad x \in \mathbb{R}. \quad (2.9)$$

The immediate corollary from the above lemma is as follows.

**Corollary 2.3.** For every  $\delta \in [1, 2]$ ,

$$\left| p_t^\alpha(x) - p_t^\alpha(y) - (x-y) \frac{\partial p_t^\alpha(y)}{\partial y} \right| \leq C \frac{|x-y|^\delta}{t^{\delta/\alpha}} (p_t^\alpha(x/2) + p_t^\alpha(y/2)), \quad t > 0, \quad x, y \in \mathbb{R}. \quad (2.10)$$

With Lemma 2.2(b) at hand, it is easy to check that if  $\beta < (\alpha - 1)/2$ , then for any  $t > 0, z \in \mathbb{R}, (s, x) \mapsto \frac{\partial p_{t-s}^\alpha(z-x)}{\partial x} \mathbf{1}_{s < t}$  is in  $\mathcal{L}_{\text{loc}}^p$  for any  $p \in (1 + \beta, \frac{1+\alpha}{2})$ . Then, using again condition (2.6), it is easy to show the following result.

**Lemma 2.4.** Let  $\beta < (\alpha - 1)/2$ . Then for any fixed  $t > 0, x \in \mathbb{R}$ , the stochastic integral

$$\int_{(0,t] \times \mathbb{R}} M(d(s, y)) \frac{\partial p_{t-s}^\alpha(y-x)}{\partial x}$$

is well defined. In what follows, we let  $\frac{\partial Z^2(x)}{\partial x}$  to denote this integral.

Let  $L = \{L_t : t \geq 0\}$  denote a spectrally positive stable process of index  $\kappa \in (1, 2)$ . That is,  $L$  is an  $\mathbb{R}$ -valued time-homogeneous process with independent increments and with Laplace transform given by

$$\mathbf{E} e^{-\lambda L_t} = e^{t\lambda^\kappa}, \quad \lambda, t \geq 0. \quad (2.11)$$

Let  $\Delta L_s := L_s - L_{s-} > 0$  denote the (positive) jumps of  $L$ . The next technical result gives an exponential upper bound for the tail of  $\sup_{0 \leq u \leq t} |L_u|$  under condition that all the jumps of  $L$  are not too large.

**Lemma 2.5.** There exists a constant  $C_{(2.12)} = C_{(2.12)}(\kappa)$  such that

$$\mathbf{P} \left( \sup_{0 \leq u \leq t} |L_u| \mathbf{1} \left\{ \sup_{0 \leq v \leq u} \Delta L_v \leq y \right\} \geq x \right) \leq \left( \frac{C_{(2.12)} t}{xy^{\kappa-1}} \right)^{x/y} \quad (2.12)$$

for all  $t, x, y > 0$ .

This bound for  $\sup_{0 \leq u \leq t} L_u$  has been proven in [4] (see Lemma 2.3 there). Since the process  $L$  is spectrally positive, the tail of  $\sup_{0 \leq u \leq t} (-L_u)$  is lighter than that of  $\sup_{0 \leq u \leq t} L_u$ . Thus, (2.12) is a consequence of [4, Lemma 2.3].

*Organization of the proof.* We would like to verify the spectrum of singularities of  $X_t(\cdot)$  on any open (random) set  $U$  whenever  $X_t(U) > 0$ . Based on the ideas of the proof of Theorem 1.1(b) in [13], it is enough to verify the spectrum of singularities of  $X_t(\cdot)$  on any fixed open ball  $U$  in  $\mathbb{R}$ . In what follows we fix, without loss of generality,  $U = (0, 1)$ . The extension of our argument to general open  $U$  is trivial.

In Section 3, we will take care of the simple terms  $Z^1$  and  $Z^3$ . In Sections 4, 5 we will verify the spectrum of singularities of  $Z^2$  which, in fact, determines the spectrum of singularities of  $X_t$ .

## 3. EASY TERMS

Consider  $Z^1, Z^3$  on the right hand side of (2.5). Clearly,  $Z^1$  is twice differentiable. Noting that  $\bar{\eta}_c < 2$  for all  $\alpha, \beta$ , we see that  $Z^1$  does not affect the optimal Hölder exponent of  $X_t$ . As for  $Z^3$ , we have the following result.

**Lemma 3.1.** *Let  $\beta < (\alpha - 1)/2$ . Then,  $\mathbf{P}$ -a.s.,  $Z^3(x)$  is differentiable for any  $x \in (0, 1)$ , and the mapping*

$$x \mapsto \frac{d}{dx} Z^3(x), \quad x \in (0, 1),$$

*is,  $\mathbf{P}$ -a.s., Hölder continuous with any exponent  $\eta < \alpha - 1$ .*

*Proof.* Using Lemma 2.12 in [4] with  $\theta = \delta = 1$ , we see that  $Z^3(x)$  is differentiable and, furthermore,

$$\frac{d}{dx} Z^3(x) = a \int_0^t ds \int_{\mathbf{R}} X_s(dy) \frac{\partial}{\partial x} p_{t-s}^\alpha(x - y).$$

Therefore, for any  $x_1, x_2 \in (0, 1)$ ,

$$\left| \frac{d}{dx} Z^3(x_1) - \frac{d}{dx} Z^3(x_2) \right| \leq |a| \int_0^t ds \int_{\mathbf{R}} X_s(dy) \left| \frac{\partial}{\partial x_1} p_{t-s}^\alpha(x_1 - y) - \frac{\partial}{\partial x_2} p_{t-s}^\alpha(x_2 - y) \right|.$$

Applying now Lemma 2.2 with  $\delta < \alpha - 1$ , we obtain

$$\begin{aligned} & \left| \frac{d}{dx} Z^3(x_1) - \frac{d}{dx} Z^3(x_2) \right| \\ & \leq C|a||x_1 - x_2|^\delta \int_0^t ds (t-s)^{-(1+\delta)/\alpha} \int_{\mathbf{R}} X_s(dy) \left( p_{t-s}^\alpha\left(\frac{x_1-y}{2}\right) + p_{t-s}^\alpha\left(\frac{x_2-y}{2}\right) \right) \\ & = C|a||x_1 - x_2|^\delta \int_0^t ds (t-s)^{-(1+\delta)/\alpha} \left( S_{2^\alpha(t-s)}^\alpha X_s(x_1) + S_{2^\alpha(t-s)}^\alpha X_s(x_2) \right) \\ & \leq C|a||x_1 - x_2|^\delta \sup_{s \leq t, x \in (0,1)} S_{2^\alpha(t-s)}^\alpha X_s(x) \int_0^t ds s^{-(1+\delta)/\alpha} \\ & = C\alpha(\alpha - 1 - \delta)^{-1} |a||x_1 - x_2|^\delta \sup_{s \leq t, x \in (0,1)} S_{2^\alpha(t-s)}^\alpha X_s(x). \end{aligned}$$

Taking into account Lemma 2.11 from [4], which states that

$$V := \sup_{s \leq t, x \in (0,1)} S_{2^\alpha(t-s)}^\alpha X_s(x) < \infty \quad \mathbf{P} - \text{a.s.}, \quad (3.1)$$

we see that  $x \mapsto \frac{d}{dx} Z^3(x)$  is Hölder continuous with the exponent  $\delta$ .  $\square$

Comining this lemma with [4, Remark 2.13], we obtain

**Corollary 3.2.**  *$\mathbb{P}$ -a.s., for any  $x \in U$  we have*

$$H_{Z^3}(x) \geq \alpha 1\{\bar{\eta}_c > 1\} + 1\{\bar{\eta}_c \leq 1\}. \quad (3.2)$$

From this corollary and the fact that the right hand side in (3.2) is not smaller than  $\bar{\eta}_c$  we conclude that  $Z^3$  does not affect the multifractal structure of  $X_t$  as well. More precisely, the spectrum of singularities of  $X_t$  coincides with that of  $Z^2$ . Consequently, to prove Theorem 1.2, we have to determine Hausdorff dimensions of the sets

$$\begin{aligned} \mathcal{E}_{Z^2, \eta} &:= \mathcal{E}_{(0,1), Z^2, \eta} = \{x \in (0, 1) : H_{Z^2}(x) = \eta\}, \\ \bar{\mathcal{E}}_{Z^2, \eta} &:= \bar{\mathcal{E}}_{(0,1), Z^2, \eta} = \{x \in (0, 1) : H_{Z^2}(x) \leq \eta\}, \end{aligned}$$

and this is done in the next two sections.

#### 4. UPPER BOUND FOR THE HAUSDORFF DIMENSION

The aim of this section is to prove the following proposition.

**Proposition 4.1.** *For every  $\eta \in [\eta_c, \bar{\eta}_c)$ ,*

$$\dim(\mathcal{E}_{Z^2, \eta}) \leq \dim(\bar{\mathcal{E}}_{Z^2, \eta}) \leq (1 + \beta)(\eta - \eta_c), \quad \mathbf{P} - \text{a.s.}$$

We need to introduce an additional notation. In what follows, for any  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ , we fix arbitrary small  $\gamma = \gamma(\eta) \in (0, \frac{10^{-2}\eta_c}{\alpha})$  such that

$$\gamma < \begin{cases} \frac{10^{-2}}{\alpha} \min\{1 - \eta, \eta\}, & \text{if } \eta < 1, \\ \frac{10^{-2}}{\alpha} \min\{\eta - 1, 2 - \eta\}, & \text{if } \eta > 1, \end{cases}$$

and define

$$S_\eta := \left\{ x \in (0, 1) : \text{there exists a sequence } (s_n, y_n) \rightarrow (t, x) \right. \\ \left. \text{with } \Delta X_{s_n}(\{y_n\}) \geq (t - s_n)^{\frac{1}{1+\beta} - \gamma} |x - y_n|^{\eta - \eta_c} \right\}.$$

To prove the above proposition we have to verify the following two lemmas.

**Lemma 4.2.** *For every  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$  we have*

$$\mathbf{P}(H_{Z^2}(x) \geq \eta - 2\alpha\gamma \text{ for all } x \in (0, 1) \setminus S_\eta) = 1.$$

**Lemma 4.3.** *For every  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$  we have*

$$\dim(S_\eta) \leq (1 + \beta)(\eta - \eta_c), \quad \mathbf{P} - \text{a.s.}$$

With Lemmas 4.2, 4.3 in hand, we immediately get

*Proof of Proposition 4.1.* It follows easily from Lemma 4.2 that  $\bar{\mathcal{E}}_{Z^2, \eta} \subset S_{\eta+2\alpha\gamma+\varepsilon}$  for every  $\varepsilon > 0$  and every  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ . Therefore,

$$\dim(\bar{\mathcal{E}}_{Z^2, \eta}) \leq \lim_{\varepsilon \rightarrow 0} \dim(S_{\eta+2\alpha\gamma+\varepsilon}).$$

Using Lemma 4.3 we then get

$$\dim(\mathcal{E}_{Z^2, \eta}) \leq (1 + \beta)(\eta + 2\alpha\gamma - \eta_c), \quad \mathbf{P} - \text{a.s.}$$

Since  $\gamma$  can be chosen arbitrary small, the result for  $\eta \neq 1$  follows immediately. The inequality for  $\eta = 1$  follows from the monotonicity in  $\eta$  of the sets  $\mathcal{E}_{Z^2, \eta}$ .  $\square$

The proof of Lemma 4.3 is rather short, so we will give it first.

*Proof of Lemma 4.3.* To every jump  $(s, y, r)$  of the measure  $\mathcal{N}$  (in what follows in the paper we will usually call them simply “jumps”) with

$$(s, r) \in D_{j, n} := [t - 2^{-j}, t - 2^{-j-1}) \times [2^{-n-1}, 2^{-n})$$

we assign the ball

$$B_{j, n}^{(s, y, r)} := B \left( y, \left( \frac{2^{-n}}{(2^{-j-1})^{\frac{1}{1+\beta} - \gamma}} \right)^{1/(\eta - \eta_c)} \right). \quad (4.1)$$

We used here the obvious notation  $B(y, \delta)$  for the ball in  $\mathbb{R}$  with the center at  $y$  and radius  $\delta$ . Define  $n_0(j) := j[\frac{1}{1+\beta} - \frac{\gamma}{4}]$ . It is clear that

$$\bigcup_{j \geq J, n \geq n_0(j)} B_{j,n}^{(s,y,r)}$$

covers  $S_\eta$  for every  $J$ . (Recall that, according to Lemma 2.14 of [4], there are no jumps bigger than  $2^{-n_0(j)}$  in the time interval  $[t - 2^{-j}, t - 2^{-j-1})$ .)

It follows from the formula for the compensator that, on  $\{\sup_{s \leq t} X_s((0, 1)) \leq N\}$ , the intensity of jumps with  $(s, r) \in D_{j,n}$  is bounded by

$$N 2^{-j-1} \int_{2^{-n-1}}^{2^{-n}} \varrho r^{-2-\beta} dr = \frac{N \varrho (2^{1+\beta} - 1)}{2(1+\beta)} 2^{n(1+\beta)-j} =: \lambda_{n,j}.$$

Therefore, the intensity of jumps with  $(s, r) \in \cup_{n=n_0(j)}^{n_1(j)} D_{j,n} := \tilde{D}_j$ , where  $n_1(j) = j[\frac{1}{1+\beta} + \frac{\gamma}{4}]$ , is bounded by

$$\sum_{n=n_0(j)}^{n_1(j)} \lambda_{j,n} \leq C 2^{j(1+\beta)\gamma/4} =: \Lambda_j$$

According to Lemma 1 from [7], the number of such jumps does not exceed  $2\Lambda_j$  with the probability  $1 - e^{-\Lambda_j}$ . Analogously, the number of jumps with  $(s, r) \in D_{j,n}$  does not exceed  $2\lambda_{j,n}$  with the probability at least  $1 - e^{-\lambda_{j,n}}$ . Since

$$\sum_j \left( e^{-\Lambda_j} + \sum_{n=n_1(j)}^{\infty} e^{-\lambda_{j,n}} \right) < \infty,$$

we conclude, applying the Borel-Cantelli lemma, that, for almost every  $\omega$  from the set  $\{\sup_{s \leq t} X_s((0, 1)) \leq N\}$ , there exists  $J(\omega)$  such that for all  $j \geq J(\omega)$  and  $n \geq n_1(j)$ , the numbers of jumps in  $\tilde{D}_j$  and in  $D_{j,n}$  are bounded by  $2\Lambda_j$  and  $2\lambda_{j,n}$  respectively.

The radius of every ball corresponding to the jump in  $\tilde{D}_j$  is bounded by  $r_j := C 2^{-\frac{3\gamma}{4(\eta-\eta_c)}j}$ . Thus, one can easily see that

$$\sum_{j=1}^{\infty} \left( 2\Lambda_j r_j^\theta + \sum_{n=n_1(j)}^{\infty} 2\lambda_{j,n} \left( \frac{2^{-n}}{(2^{-j-1})^{\frac{1}{1+\beta}-\gamma}} \right)^{\theta/(\eta-\eta_c)} \right) < \infty$$

for every  $\theta > (1+\beta)(\eta-\eta_c)$ . This yields the desired bound for the Hausdorff dimension for almost every  $\omega \in \{\sup_{s \leq t} X_s((0, 1)) \leq N\}$ . Letting  $N \rightarrow \infty$  completes the proof.  $\square$

The remaining part of this section will be devoted to the proof of Lemma 4.2. Let

$$S_\eta(J) = \bigcap_{j \geq J} \bigcup_{n \geq n_0(j)} \bigcup_{D_{j,n}} B_{(s,y,r)}.$$

Since  $S_\eta = \cap_{J \geq 1} S_\eta(J)$ ,

$$\{H_{Z^2}(x) \geq \eta - 2\alpha\gamma, x \in (0, 1) \setminus S_\eta\} = \bigcap_{J \geq 1} \{H_{Z^2}(x) \geq \eta - 2\alpha\gamma, x \in (0, 1) \setminus S_\eta(J)\}.$$



Thus, it suffices to show that

$$\mathbf{P}(H_{Z^2}(x) \geq \eta - 2\alpha\gamma, \forall x \in (0, 1) \setminus S_\eta(J)) = 1 \quad (4.2)$$

for every  $J \geq 1$ .

Before we start proving (4.2), let us introduce some further notation. For any  $x_1, x_2 \in \mathbb{R}, \eta \in (\eta_c, \bar{\eta}_c)$  define

$$\begin{aligned} \tilde{p}_s^{\alpha, \eta}(x, y) &\equiv \begin{cases} p_s^\alpha(x) - p_s^\alpha(y), & \text{if } \eta \leq 1, s > 0 \\ p_s^\alpha(x) - p_s^\alpha(y) - (x - y) \frac{\partial p_s^\alpha(y)}{\partial y}, & \text{if } \eta \in (1, \bar{\eta}_c), s > 0, \end{cases} \\ \tilde{p}_s^{\alpha, \eta'}(x, y) &\equiv \frac{\partial p_s^\alpha(x)}{\partial x} - \frac{\partial p_s^\alpha(y)}{\partial y}, \quad \eta \in (1, \bar{\eta}_c), \\ \tilde{Z}_s^{2, \eta}(x_1, x_2) &\equiv \int_0^s \int_{\mathbb{R}} M(d(u, y)) \tilde{p}_{t-u}^{\alpha, \eta}(x_1 - y, x_2 - y), \quad s \in [0, t], \\ \Delta \tilde{Z}_s^{2, \eta}(x_1, x_2) &\equiv \tilde{Z}_s^{2, \eta}(x_1, x_2) - \tilde{Z}_{s-}^{2, \eta}(x_1, x_2), \quad s \in (0, t], \\ \tilde{Z}_s^{2, \eta'}(x_1, x_2) &\equiv \int_0^s \int_{\mathbb{R}} M(d(u, y)) \tilde{p}_{t-u}^{\alpha, \eta'}(x_1 - y, x_2 - y), \quad s \in [0, t], \\ \Delta \tilde{Z}_s^{2, \eta'}(x_1, x_2) &\equiv \tilde{Z}_s^{2, \eta'}(x_1, x_2) - \tilde{Z}_{s-}^{2, \eta'}(x_1, x_2), \quad \eta \in (1, \bar{\eta}_c), s \in (0, t], \end{aligned}$$

Also for any  $N, J \geq 1$ , let

$$\tilde{S}_\eta(N, J) := \{(x_1, x_2) \in \mathbb{R}^2 : \exists x_0 \in (0, 1) \setminus S_\eta(J) \text{ such that } x_1, x_2 \in B(x_0, 2^{-N})\}$$

and

$$\tilde{S}'_\eta(J) := \{(x_1, x_2) \in \mathbb{R}^2 : \exists x_0 \in (0, 1) \setminus S_\eta(J) \text{ such that } x_1, x_2 \in B(x_0, 4|x_1 - x_2|)\}.$$

We also have to introduce a “good” event  $A^\varepsilon$ , on which, with high probability,  $V$  from (3.1) is bounded by a constant, and there is a bound on sizes of jumps. Fix some  $\varepsilon \in (0, \eta_c/2)$  arbitrary small. Let  $\Delta X_s = X_s - X_{s-}$  denote the jumps of the measure-valued process  $X$ . By Lemma 2.14 of [4], there exists a constant  $C_{(4.3)} = c_{(4.3)}(\varepsilon, \gamma)$  such that

$$\mathbf{P}(|\Delta X_s| > C_{(4.3)}(t - s)^{(1+\beta)^{-1} - \gamma} \text{ for some } s < t) \leq \varepsilon/10. \quad (4.3)$$

Then we fix another constant  $C_{(4.4)} = C_{(4.4)}(\varepsilon, \gamma)$  such that

$$\mathbf{P}(V \leq C_{(4.4)}) \geq 1 - \varepsilon/10. \quad (4.4)$$

Recall that, by Theorem 1.2 in [4],  $x \mapsto X_t(x)$  is  $\mathbf{P}$ -a.s. Hölder continuous with any exponent less than  $\eta_c$ . Hence we can define a constant  $C_{(4.5)} = C_{(4.5)}(\varepsilon)$  such that

$$\mathbf{P}\left(\sup_{x_1, x_2 \in (0, 1), x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^{\eta_c - \varepsilon}} \leq C_{(4.5)}\right) \geq 1 - \varepsilon/10. \quad (4.5)$$

Then define

$$A^\varepsilon \equiv \{|\Delta X_s| \leq C_{(4.3)}(t - s)^{(1+\beta)^{-1} - \gamma} \text{ for all } s < t\} \quad (4.6)$$

$$\cap \{V \leq C_{(4.4)}\} \cap \left\{ \sup_{x_1, x_2 \in (0, 1), x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^{\eta_c - \varepsilon}} \leq C_{(4.5)} \right\}.$$

Clearly by (4.3) and (4.4),  $\mathbf{P}(A^\varepsilon) \geq 1 - \varepsilon$ . See (3.4) in [4] for the analogous definition.

We split the proof of (4.2) into several steps.

**Lemma 4.4.** *Fix arbitrary (deterministic)  $x_1, x_2 \in \mathbb{R}$ , and  $\eta \in (\eta_c, \bar{\eta}_c)$ . Then for any  $N, J \geq 1$ , there exists a constant  $C_{(4.7)}(J) \geq 1$  such that*

$$\begin{aligned} & \left| \Delta \tilde{Z}_s^{2,\eta}(x_1, x_2) \right| 1_{A^\varepsilon} 1_{\{(x_1, x_2) \in \tilde{S}_\eta(N, J)\}} \\ & \leq C_{(4.7)}(J) |x_1 - x_2|^{\eta_c - \alpha\gamma} 2^{-N(\eta - \eta_c)}, \quad \forall s \leq t. \end{aligned} \quad (4.7)$$

*Proof.* Let  $(y, s, r)$  be the point of arbitrary jump of the measure  $\mathcal{N}$  with  $s \leq t$ . Then for corresponding jump of  $\tilde{Z}_s^{2,\eta}(x_1, x_2)$  we get the following bound

$$\left| \Delta \tilde{Z}_s^{2,\eta}(x_1, x_2) \right| \leq r \left| \tilde{p}_{t-s}^{\alpha,\eta}(x_1 - y, x_2 - y) \right|. \quad (4.8)$$

Now on the event  $\{(x_1, x_2) \in \tilde{S}_\eta(N, J)\}$ , there exists a point  $x_0 \in (0, 1) \setminus S_\eta(J)$  such that  $x_1, x_2 \in B(x_0, 2^{-N})$  and for  $s \geq t - 2^{-J}$  we have

$$r \leq (t - s)^{\frac{1}{1+\beta} - \gamma} |y - x_0|^{\eta - \eta_c}.$$

This and (4.8) imply that for  $s \geq t - 2^{-J}$

$$\begin{aligned} I &= I(s, y, x_1, x_2) \equiv \left| \Delta \tilde{Z}_s^{2,\eta}(x_1, x_2) \right| 1_{A^\varepsilon} 1_{\{(x_1, x_2) \in \tilde{S}_\eta(N, J)\}} \\ &\leq (t - s)^{\frac{1}{1+\beta} - \gamma} |y - x_0|^{\eta - \eta_c} \left| \tilde{p}_{t-s}^{\alpha,\eta}(x_1 - y, x_2 - y) \right|. \end{aligned} \quad (4.9)$$

Applying Lemma 2.1 (if  $\eta \leq 1$ ) or Corollary 2.3 (if  $\eta > 1$ ) with  $\delta = \eta - \alpha\gamma$  to  $\tilde{p}_{t-s}^{\alpha,\eta}(x_1 - y, x_2 - y)$ , we conclude from (4.9) that, for  $s \geq t - 2^{-J}$ ,

$$\begin{aligned} I &\leq (t - s)^{-(\eta - \eta_c)/\alpha} |y - x_0|^{\eta - \eta_c} |x_1 - x_2|^{\eta - \alpha\gamma} \\ &\quad \times \left( p_1^\alpha \left( \frac{|x_1 - y|}{2(t - s)^{1/\alpha}} \right) + p_1^\alpha \left( \frac{|x_2 - y|}{2(t - s)^{1/\alpha}} \right) \right) \\ &\leq C_{(4.10)} (t - s)^{-(\eta - \eta_c)/\alpha} |y - x_0|^{\eta - \eta_c} |x_1 - x_2|^{\eta - \alpha\gamma} \\ &\quad \times \left( \frac{|x_1 - y| + |x_2 - y|}{(t - s)^{1/\alpha}} \vee 1 \right)^{-\alpha - 1}, \end{aligned} \quad (4.10)$$

where the last inequality follows from the standard bound

$$p_1^\alpha(z) \leq C_{(4.11)} (|z| \vee 1)^{-\alpha - 1}, \quad z \in \mathbb{R}, \quad (4.11)$$

One can easily check by separating the cases  $|x_1 - y| + |x_2 - y| < (t - s)^{1/\alpha}$  and  $|x_1 - y| + |x_2 - y| \geq (t - s)^{1/\alpha}$  that

$$|x_1 - x_2|^{\eta - \eta_c} \left( \frac{|x_1 - y| + |x_2 - y|}{(t - s)^{1/\alpha}} \vee 1 \right)^{-\alpha - 1} \leq (t - s)^{(\eta - \eta_c)/\alpha},$$

and hence

$$I \leq C_{(4.10)} |x_1 - x_2|^{\eta_c - \alpha\gamma} |y - x_0|^{\eta - \eta_c}, \quad s \geq t - 2^{-J}. \quad (4.12)$$

If  $|y - x_0| \leq 2^{-N+1}$ , then we obtain the bound

$$I \leq 2C_{(4.10)} |x_1 - x_2|^{\eta_c - \alpha\gamma} 2^{-N(\eta - \eta_c)}, \quad s \geq t - 2^{-J}. \quad (4.13)$$

Now consider the case  $|y - x_0| > 2^{-N+1}$ . Here we treat separately two subcases:  $|y - x_0| \leq (t - s)^{1/\alpha}$  and  $|y - x_0| > (t - s)^{1/\alpha}$ . First, if  $|y - x_0| \leq (t - s)^{1/\alpha}$ , then

it follows from (4.10) that

$$\begin{aligned} I &\leq C_{(4.10)}(t-s)^{-(\eta-\eta_c)/\alpha}|y-x_0|^{\eta-\eta_c}|x_1-x_2|^{\eta-\alpha\gamma} \\ &\leq C_{(4.10)}|x_1-x_2|^{\eta-\alpha\gamma} \\ &\leq C_{(4.10)}|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-(N-1)(\eta-\eta_c)}, \quad s \geq t-2^{-J}. \end{aligned} \quad (4.14)$$

Second, if  $|y-x_0| > (t-s)^{1/\alpha}$ , then we recall that  $|y-x_0| > (t-s)^{1/\alpha} \vee 2^{-N+1}$  and  $|x_i-x_0| \leq 2^{-N}$ ,  $i=1,2$ , to get that

$$|x_i-y| \geq |x_0-y|/2, \quad i=1,2. \quad (4.15)$$

Combining this with (4.10), we obtain

$$\begin{aligned} I &\leq C_{(4.10)}(t-s)^{-(\eta-\eta_c)/\alpha}|y-x_0|^{\eta-\eta_c}|x_1-x_2|^{\eta-\alpha\gamma} \left( \frac{|x_0-y|}{(t-s)^{1/\alpha}} \right)^{-\alpha-1} \\ &\leq C_{(4.10)}|x_1-x_2|^{\eta-\alpha\gamma} \left( \frac{|x_0-y|}{(t-s)^{1/\alpha}} \right)^{\eta-\eta_c-\alpha-1} \\ &\leq C_{(4.10)}|x_1-x_2|^{\eta-\alpha\gamma} \\ &\leq C_{(4.10)}|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-(N-1)(\eta-\eta_c)}, \quad s \geq t-2^{-J}. \end{aligned} \quad (4.16)$$

Finally, we consider the jumps  $(y, s, r)$  with  $s < t-2^{-J}$ . On the event  $A^\varepsilon$ ,

$$r \leq (t-s)^{\frac{1}{1+\beta}-\gamma}.$$

Using Lemma 2.1 (or Corollary 2.3) with  $\delta = \eta - \alpha\gamma$  once again, we see from (4.8) that

$$\begin{aligned} I &\leq C_{(4.17)}|x_1-x_2|^{\eta-\alpha\gamma}(t-s)^{-(\eta-\eta_c)/\alpha} \\ &\leq C_{(4.17)}2^{J(\eta-\eta_c)/\alpha}|x_1-x_2|^{\eta-\alpha\gamma} \\ &\leq C_{(4.17)}2^{J(\eta-\eta_c)/\alpha}|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-(N-1)(\eta-\eta_c)}, \quad s < t-2^{-N}. \end{aligned} \quad (4.17)$$

Combining (4.13) – (4.17) we get the desired result.  $\square$

By similar argument we can get the following result.

**Lemma 4.5.** *Let  $\bar{\eta}_c > 1$ . Fix arbitrary (deterministic)  $x_1, x_2 \in \mathbb{R}$ , and  $\eta \in (1, \bar{\eta}_c)$ . Then for any  $J \geq 1$ , there exists a constant  $C_{(4.18)}(J)$  such that*

$$\begin{aligned} &\left| \Delta \tilde{Z}_s^{\eta, 2, J}(x_1, x_2) \right| 1_{A^\varepsilon} 1_{\{(x_1, x_2) \in \tilde{S}_\eta(J)\}} \\ &\leq C_{(4.18)}(J)|x_1-x_2|^{\eta-1-\alpha\gamma}, \quad \forall s \leq t. \end{aligned} \quad (4.18)$$

Having an upper bound for absolute values of the jumps of  $\tilde{Z}^2(x_1, x_2)$ , we can give some estimate for  $\tilde{Z}_t^2(x_1, x_2)$  itself:

**Lemma 4.6.** *Fix arbitrary (deterministic)  $x_1, x_2 \in (0, 1)$ , and  $\eta \in (\eta_c, \bar{\eta}_c)$ .*

(a) *Then there exists a constant  $C_{(4.19)}$ , such that for any  $N, J \geq 1$ ,*

$$\begin{aligned} \mathbf{P} \left( |\tilde{Z}_t^{2, \eta}(x_1, x_2)| \geq 2C_{(4.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, (x_1, x_2) \in \tilde{S}_\eta(N, J) \right) \\ \leq \left( C_{(4.19)}2^{-\alpha\gamma N} \right)^{|x_1-x_2|^{-\alpha\gamma}}. \end{aligned} \quad (4.19)$$

(b) Let  $\bar{\eta}_c > 1$  and assume  $\eta \in (1, \bar{\eta}_c)$ . Then there exists a constant  $C_{(4.20)}$ , such that for any  $J \geq 1$ ,

$$\begin{aligned} \mathbf{P} \left( |\tilde{Z}_t^{2,\eta'}(x_1, x_2)| \geq 2C_{(4.18)}(J)|x_1 - x_2|^{\eta-1-2\alpha\gamma}, A^\varepsilon, (x_1, x_2) \in \tilde{S}'_\eta(J) \right) \\ \leq \left( C_{(4.20)}|x_1 - x_2|^{\alpha\gamma} \right)^{|x_1 - x_2|^{-\alpha\gamma}}. \end{aligned} \quad (4.20)$$

*Proof.* (a) According to Lemma 2.15 from [4] there exist spectrally positive  $(1+\beta)$ -stable processes  $L^+$  and  $L^-$  such that

$$\tilde{Z}_s^{2,\eta}(x_1, x_2) = L_{T_+^\eta(s)}^+ - L_{T_-^\eta(s)}^-, \quad s \leq t, \quad (4.21)$$

where

$$T_\pm^\eta(s) = \int_0^s du \int_{\mathbb{R}} X_u(dy) \left( (\tilde{p}_{t-u}^{\alpha,\eta}(x_1 - y, x_2 - y))^\pm \right)^{1+\beta}, \quad 0 \leq s \leq t. \quad (4.22)$$

(Note that  $L^+, L^-$  also depend on  $\eta$  however we omit the corresponding superindex  $\eta$  to simplify the notation.) Therefore we get

$$\begin{aligned} \mathbf{P} \left( |\tilde{Z}_t^{2,\eta}(x_1, x_2)| \geq 2C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, (x_1, x_2) \in \tilde{S}_\eta(N, J) \right) \\ \leq \mathbf{P} \left( |L_{T_+^\eta(s)}^+| \geq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, (x_1, x_2) \in \tilde{S}_\eta(N, J) \right) \\ + \mathbf{P} \left( |L_{T_-^\eta(s)}^-| \geq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, (x_1, x_2) \in \tilde{S}_\eta(N, J) \right). \end{aligned} \quad (4.23)$$

By going through the derivation of (3.43) in [5], one can easily get that in our setting, on the event  $A^\varepsilon$ , for any  $\eta \in (\eta_c, \bar{\eta}_c)$  and any  $\varepsilon_1 \in (0, \alpha\beta\gamma)$ , there exists a constant  $C_{(4.24)} = C_{(4.24)}(\varepsilon, \varepsilon_1, \eta)$  such that

$$T_\pm^\eta(t) \leq C_{(4.24)}|x_1 - x_2|^{\alpha-\beta-\varepsilon_1} =: \hat{T}(x_1, x_2). \quad (4.24)$$

From this bound, Lemmas 2.5 and 4.4 we get

$$\begin{aligned} \mathbf{P} \left( |L_{T_\pm^\eta(t)}^\pm| \geq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, (x_1, x_2) \in \tilde{S}_\eta(N, J) \right) \\ \leq \mathbf{P} \left( |L_{T_\pm^\eta(t)}^\pm| \geq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, \right. \\ \left. \sup_{s \leq T_\pm^\eta} \Delta L_s^\pm \leq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-\alpha\gamma}2^{-N(\eta-\eta_c)} \right) \\ \leq \mathbf{P} \left( \sup_{0 \leq s \leq \hat{T}(x_1, x_2)} |L_s^\pm| 1_{\left\{ \sup_{0 \leq v \leq s} \Delta L_v^\pm \leq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-\alpha\gamma}2^{-N(\eta-\eta_c)} \right\}} \right. \\ \left. \geq C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)} \right) \\ \leq \left( \frac{C_{(2.12)}|x_1 - x_2|^{\alpha-\beta-\varepsilon_1}}{|x_1 - x_2|^{-\alpha\gamma}(C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-\alpha\gamma}2^{-N(\eta-\eta_c)})^{1+\beta}} \right)^{|x_1 - x_2|^{-\alpha\gamma}} \\ \leq \left( \frac{C_{(2.12)}|x_1 - x_2|^{\alpha\gamma(2+\beta)-\varepsilon_1+1}}{2^{-N(\eta-\eta_c)(1+\beta)}} \right)^{|x_1 - x_2|^{-\alpha\gamma}} \\ \leq \left( 4C_{(2.12)}2^{-\alpha\gamma N} \right)^{|x_1 - x_2|^{-\alpha\gamma}}, \end{aligned}$$

where the last inequality follows since  $(\eta - \eta_c)(1 + \beta) \leq 1, \varepsilon_1 \leq \alpha\beta\gamma, |x_1 - x_2| \leq 2^{-N+1}$ , and  $C_{(4.7)}(J) \geq 1$ . (We omit here some elementary arithmetic calculations.)

The claim follows now from (4.23).

(b) The proof goes along the similar lines.  $\square$

**Lemma 4.7.** *Let  $J \geq 1, \bar{\eta}_c > 1, \eta \in (1, \bar{\eta}_c)$ . For almost every  $\omega \in A^\varepsilon$  there exists  $N_2 = N_2(\omega)$  such that for all  $n \geq N_2$ , and*

$$(x_1, x_2) \in \tilde{S}'_\eta(J) \cap \{(i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}, |i - j| = 1\}$$

the following holds

$$|\tilde{Z}_t^{2,\eta'}(x_1, x_2)| \leq 2C_{(4.18)}(J)|x_1 - x_2|^{\eta-1-2\alpha\gamma}.$$

*Proof.* Define

$$M_n := \max \left\{ |\tilde{Z}_t^{2,\eta'}(x_1, x_2)| : (x_1, x_2) \in \{(i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}, |i - j| = 1\} \cap (0, 1) \cap \tilde{S}'_\eta(J) \right\}.$$

Applying Lemma 4.6(b), we obtain

$$\mathbf{P} \left( M_n \geq 2C_{(4.18)}(J)2^{-n(\eta-1-2\alpha\gamma)}; A^\varepsilon \right) \leq 2^n \left( C_{(4.20)}2^{-n\alpha\gamma} \right)^{2^{n\alpha\gamma}}.$$

Let

$$A_N := \{M_n \geq 2C_{(4.18)}(J)2^{-n(\eta-1-2\alpha\gamma)} \text{ for some } n \geq N\}.$$

It is clear that

$$\begin{aligned} \sum_{N=1}^{\infty} \mathbf{P}(A_N \cap A^\varepsilon) &\leq \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} \mathbf{P} \left( M_n \geq 2C_{(4.18)}(J)2^{-n(\eta-1-2\alpha\gamma)}; A^\varepsilon \right) \\ &\leq \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} 2^n \left( C_{(4.20)}2^{-n\alpha\gamma} \right)^{2^{n\alpha\gamma}} \\ &\leq \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} 2^n \left( C_{(4.20)}2^{-n\alpha\gamma} \right)^{2^{n\alpha\gamma}} \\ &\leq C \sum_{N=1}^{\infty} 2^{-\alpha\gamma N} < \infty \end{aligned}$$

and we are done by the Borel-Cantelli lemma.  $\square$

**Lemma 4.8.** *Let  $J \geq 1, \eta \in (\eta, \bar{\eta}_c)$ . For almost every  $\omega \in A^\varepsilon$  there exists  $N_1 = N_1(\omega)$  such that for all  $n \geq N \geq N_1$ ,*

$$(x_1, x_2) \in \tilde{S}_\eta(N, J) \cap \{(i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}\}$$

with  $|x_1 - x_2| \leq 2^{-\log^2 n}$  we have the inequality

$$|\tilde{Z}_t^{2,\eta}(x_1, x_2)| \leq 2C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}.$$

*Proof.* Define

$$A_{n,N} := \bigcup_{\substack{x_1, x_2 \in \{i2^{-n}, i \in \mathbb{Z}\} \\ \cap(0, 1) \cap |\tilde{S}_\eta(N, J)| : \\ \{|x_1 - x_2| \leq 2^{-\log^2 n}\}}} \left\{ |\tilde{Z}_t^{2,\eta}(x_1, x_2)| \geq 2C_{(4.7)}(J)|x_1 - x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)} \right\}.$$

Applying Lemma 4.6(a), we obtain

$$\mathbf{P}\left(A_{n,N} \cap A^\varepsilon\right) \leq 2^{2n} \left((C_{4.19})2^{-\alpha\gamma N}\right)^{2^{\alpha\gamma \log^2 n}}.$$

Let

$$A_N := \bigcup_{n \geq N} A_{n,N}.$$

It is clear that

$$\begin{aligned} \sum_{N=1}^{\infty} \mathbf{P}(A_N \cap A^\varepsilon) &\leq \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} \mathbf{P}\left(A_{n,N} \cap A^\varepsilon\right) \\ &\leq \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} 2^{2n} \left((C_{4.19})2^{-\alpha\gamma N}\right)^{2^{\alpha\gamma \log^2 n}} \\ &\leq C \sum_{N=1}^{\infty} 2^{-\alpha\gamma N} < \infty \end{aligned}$$

and we are done by the Borel-Cantelli lemma.  $\square$

**Lemma 4.9.** *Let  $J \geq 1$ ,  $\bar{\eta}_c > 1$ ,  $\eta \in (1, \bar{\eta}_c)$ . Fix an integer  $k_0 > \max\left\{1 + \frac{1}{\alpha-1-\beta}, 3\right\}$ . For almost every  $\omega \in A^\varepsilon$  and for all  $x \in (0, 1) \setminus S_\eta(J)$ , there exists  $V'(x) = V'(x, \omega)$  and  $C_{(4.25)}(J, \omega)$  such that*

$$|Z^2(y) - Z^2(x) - (y - x)V'(x)| \leq C_{(4.25)}(J)|y - x|^{\eta-2\alpha\gamma}, \quad \forall y \in B(x, 2^{-N_3}), \quad (4.25)$$

where

$$N_3 = \max\{N_2(\omega), N_1(\omega), \log_2(|V'(x)|), (k_0)^{10}\} + 2$$

and  $N_2, N_1$  are from Lemmas 4.7, 4.8.

**Remark 4.10.**  $Z^2(x)$  (similarly for  $Z^2(y)$ ) at a random point  $x$  is defined via  $Z^2(x) = X_t(x) - Z^1(x) - Z^3(x)$ , where all the terms on the right hand side are well defined.

**Remark 4.11.** The lemma shows that  $V'(x)$  is in fact a spatial derivative of  $Z^2(x)$  at point  $x$ .

*Proof.* First we will define  $V'(y)$  for fixed points  $y$ . For any  $y \in \mathbb{R}$ , let

$$V'(y) \equiv \int_0^t \int_{\mathbb{R}} M(d(u, z)) \frac{\partial}{\partial y} p_{t-u}^\alpha(y - z).$$

Let  $x$  and  $\omega$  be as in the statement of the lemma. For any  $n \geq 1$ , take  $x_n \in \{i2^{-n}, i \in \mathbb{Z}\}$  satisfying the following conditions

$$|x_n - x| \leq 2^{-n}, \quad |x_n - x_{n+1}| = 2^{-n-1}, \quad \forall n \geq 1. \quad (4.26)$$

Applying Lemma 4.7, we get for every  $n \geq N_3$  the bound

$$\begin{aligned} |V'(x_n) - V'(x_{n+1})| &= |\tilde{Z}_t^{2,\eta'}(x_n, x_{n+1})| \\ &\leq 2C_{(4.18)}(J)2^{-n(\eta-1-2\alpha\gamma)}. \end{aligned}$$

Then for any  $m > n \geq N_3$  we have

$$\begin{aligned} |V'(x_n) - V'(x_m)| &\leq \sum_{k=n}^{m-1} |V'(x_k) - V'(x_{k+1})| \\ &\leq C_{(4.27)}(J)2^{-n(\eta-1-2\alpha\gamma)}. \end{aligned} \quad (4.27)$$

This implies that  $\{V'(x_n)\}_{n \geq 1}$  is a Cauchy sequence and we denote the limit by  $V'(x)$ . Moreover it is easy to check that

$$|V'(x_n) - V'(x)| \leq C_{(4.27)}(J)2^{-n(\eta-1-2\alpha\gamma)}, \quad n \geq N_3. \quad (4.28)$$

Now let us check (4.25). Let

$$y \in B(x, 2^{-N_3}) \setminus \{x\}.$$

Then we can fix an integer  $N^* \geq N_3$  such that

$$2^{-N^*-1} \leq |x - y| \leq 2^{-N^*}. \quad (4.29)$$

Fix a sequence  $\{x_n\}_{n \geq 1}$  satisfying (4.26), and  $\{y_n\}_{n \geq 1}$  satisfying the same condition with  $y$  instead of  $x$ . Then for any  $n \geq N_3$  we have

$$\begin{aligned} &|Z^2(y) - Z^2(x) - (y - x)V'(x)| \\ &\leq |Z^2(y_n) - Z^2(x_n) - (y_n - x_n)V'(x_n)| \\ &\quad + |Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)| \\ &\quad + |y_n - y| \times |V'(x)| + |x_n - x| \times |V'(x)| \\ &\quad + |y_n - x_n| \times |V'(x) - V'(x_n)|. \end{aligned} \quad (4.30)$$

In what follows fix

$$n = k_0 N^*. \quad (4.31)$$

Then we have

$$\begin{aligned} &|y_n - y| \times |V'(x)| + |x_n - x| \times |V'(x)| \\ &\leq 2 \cdot 2^{-k_0 N^*} |V'(x)| \\ &\leq 2 \cdot 2^{-N^*(\eta-2\alpha\gamma)} 2^{-N^*} |V'(x)| \quad (\text{since } \eta \leq 2, k_0 \geq 3) \\ &\leq (2|x - y|)^{\eta-2\alpha\gamma} 2^{1-N^*} |V'(x)| \quad (\text{by (4.29)}) \\ &\leq (|x - y|)^{\eta-2\alpha\gamma} 2^{2-N^*} |V'(x)| \\ &\leq |x - y|^{\eta-2\alpha\gamma}, \forall n \geq N'_1, \end{aligned} \quad (4.32)$$

where the last inequality follows from  $N^* \geq \log_2(|V'(x)|) + 2$ . Now by triangle inequality and (4.31) we get

$$\begin{aligned} |x_n - y_n| &\leq 2^{1-n} + |x - y| \\ &\leq 2^{1-N^*} \\ &\leq 4|x - y|. \end{aligned} \quad (4.33)$$

This, (4.31) and (4.28) imply

$$\begin{aligned} |y_n - x_n| \cdot |V'(x) - V'(x_n)| &\leq 4|x - y| \cdot C_{(4.27)}(J) 2^{-N^* k_0 (\eta - 1 - 2\alpha\gamma)} \\ &\leq 8C(J) |x - y| \cdot |x - y|^{\eta - 1 - 2\alpha\gamma} \\ &\leq 8C(J) |x - y|^{\eta - 2\alpha\gamma}. \end{aligned} \quad (4.34)$$

Now recall that  $Z^2$  is Hölder continuous with any exponent less than  $\eta_c$  (see Theorem 2 in [4]) to get that there exists  $C = C(\omega)$  such that

$$\begin{aligned} &|Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)| \\ &\leq C(\omega) \left( |y_n - y|^{\eta_c - (2\alpha\gamma)/k_0} + |x_n - x|^{\eta_c - (2\alpha\gamma)/k_0} \right), \forall y \in B(x, 2^{-N_3}). \end{aligned}$$

Recalling that

$$|y_n - y|, |x_n - x| \leq 2^{-n} \quad (4.35)$$

and (4.31) we get

$$\begin{aligned} |Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)| &\leq 2C(\omega) 2^{-n(\eta_c - (2\alpha\gamma)/k_0)} \\ &\leq 2C(\omega) 2^{-N^*(k_0\eta_c - 2\alpha\gamma)} \\ &\leq 2C(\omega) 2^{-N^*(\eta - 2\alpha\gamma)}, \end{aligned}$$

where the last inequality follows since by assumption  $k_0 > 1 + 1/(\alpha - 1 - \beta)$  and hence  $k_0\eta_c > \bar{\eta}_c > \eta$ . By (4.29) we immediately get

$$|Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)| \leq 8C(\omega) |x - y|^{\eta - 2\alpha\gamma}. \quad (4.36)$$

Now use again (4.35) and triangle inequality to get that

$$|y_n - x| \leq |y_n - y| + |y - x| \leq 2^{-(N^* - 1)}.$$

This together with the (4.35) and the definition of  $\tilde{S}_\eta(N, J)$  implies that

$$(x_n, y_n) \in \tilde{S}_\eta(N^* - 1, J) \cap \{(i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}\}. \quad (4.37)$$

Note that

$$\begin{aligned} |x_n - y_n| &\leq 2^{-(N^* - 1)} \quad (\text{by (4.33)}) \\ &\leq 2^{-\log^2(k_0 N^*)} \\ &= 2^{-\log^2(n)}, \end{aligned} \quad (4.38)$$

where the first inequality follows by (4.33) and the second inequality follows easily by our assumption  $N^* \geq (k_0)^{10}$ ,  $k_0 > 3$ . By (4.37), (4.38) and since  $N^* - 1 \geq N_1$ , we can apply Lemma 4.8 to get

$$\begin{aligned} &|Z^2(y_n) - Z^2(x_n) - (y_n - x_n)V'(x_n)| \\ &= |Z_t^{2,\eta}(x_n, y_n)| \\ &\leq C|y_n - x_n|^{\eta - 2\alpha\gamma} \\ &\leq C'|y - x|^{\eta - 2\alpha\gamma}, \end{aligned} \quad (4.39)$$

where the last inequality follows by (4.33).

By (4.30) and the bounds (4.32), (4.34), (4.36), (4.39) we complete the proof.  $\square$



**Lemma 4.12.** *Let  $J \geq 1$ ,  $\eta \in (\eta_c, \min\{\bar{\eta}_c, 1\})$ . For almost every  $\omega \in A^\varepsilon$  and for all  $x \in (0, 1) \setminus S_\eta(J)$ ,*

$$|Z^2(y) - Z^2(x)| \leq C_1(J)|y - x|^{\eta - 2\alpha\gamma}, \quad \forall y \in B(x, 2^{-N_1}),$$

where  $N_1 = N_1(\omega)$  is from Lemma 4.8.

*Proof.* For any  $n \geq 1$ , take  $x_n, y_n \in \{i2^{-n}, i \in \mathbb{Z}\}$  satisfying the following conditions

$$\begin{aligned} |x_n - x| &\leq 2^{-n}, \quad |x_n - x_{n+1}| \leq 2^{-n-1}, \\ |y_n - y| &\leq 2^{-n}, \quad |y_n - y_{n+1}| \leq 2^{-n-1}. \end{aligned}$$

Applying Lemma 4.8, we get for every  $N \geq N_1$  the bound

$$\begin{aligned} &|Z_t^{2,\eta}(y, x)| \\ &\leq |Z_t^{2,\eta}(y_N, x_N)| + \sum_{n=N}^{\infty} \left( |Z_t^{2,\eta}(y_{N+1}, y_N)| + |Z_t^{2,\eta}(x_{N+1}, x_N)| \right) \\ &\leq 2C(J)2^{-N(\eta - \eta_c)} \left( |x_N - y_N|^{\eta_c - 2\alpha\gamma} + 2 \sum_{n=N}^{\infty} 2^{-n(\eta_c - \alpha\gamma)} \right) \\ &\leq C'(J)2^{-N(\eta - \eta_c)} \left( |x_N - y_N|^{\eta_c - 2\alpha\gamma} + 2^{-N(\eta_c - 2\alpha\gamma)} \right). \end{aligned} \quad (4.40)$$

Choosing  $N$  so that  $|y - x| \in [2^{-N-1}, 2^{-N}]$ , we finish the proof.  $\square$

Now we are able to complete

*Proof of Lemma 4.2.* Lemmas 4.9, 4.12 imply that

$$\mathbf{P}(H_{Z^2}(x) \geq \eta - 2\alpha\gamma, \forall x \in (0, 1) \setminus S_\eta(J); A^\varepsilon) \geq 1 - \varepsilon.$$

Letting here  $\varepsilon \rightarrow 0$  we complete the proof of the lemma.  $\square$

## 5. LOWER BOUND FOR THE HAUSDROFF DIMENSION.

The aim of this section is to prove the following proposition.

**Proposition 5.1.** *For every  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ ,*

$$\dim(\mathcal{E}_{Z^2, \eta}) \geq (1 + \beta)(\eta - \eta_c), \quad \mathbf{P} - \text{a.s. on } \{X_t((0, 1)) > 0\}.$$

**Remark 5.2.** Clearly the above proposition together with Proposition 4.1 finishes the proof of Theorem 1.2.

Before we begin with the proof of the proposition, we notice that the lower bound is much more involved than the upper one. And this not unexpected: in our previous papers [4, 5] the proofs of the optimality of Hölder indices were harder than the derivation of the Hölder continuity. It is similar, in some sense, to large deviation problems, where an upper bound can usually be obtained from some analytical estimates (for example, exponential Chebyshev inequality), but for a lower bound one has to understand the 'optimal strategy' of a stochastic process under consideration. Due to the mentioned complexity of the proof we give, for reader's convenience, a short description of our strategy. After auxiliary lemmata we construct a set  $\tilde{V}_\eta$  with  $\dim(\tilde{V}_\eta) \geq (\beta + 1)(\eta - \eta_c)$ , on which we show existence of "big" jumps of  $X$  that occur close to time  $t$ . These jumps are "encoded" in the jumps of the auxiliary processes  $L^+$  and they, in fact, destroy the Hölder continuity on  $\tilde{V}_\eta$  of any index greater or equal to  $\eta$  (see Lemma 5.7 and Lemma 5.8). However,

there are also other jumps of the process  $X$  (they will be encoded in processes  $L^-$ ) which may compensate the impact of the jumps of  $X$  encoded in  $L^+$ . The most difficult part of the proof is to show that there is no such compensation. More precisely, we prove that such a compensation is possible on a set of the Hausdorff dimension strictly smaller than  $(\beta + 1)(\eta - \eta_c)$ , and hence does not influence the dimension result. It is done in Lemmata 5.9, 5.10 and 5.12.

We start with deriving uniform estimates for  $X_s(I_k^{(n)})$ , where

$$I_k^{(n)} := [k2^{-n}, (k+1)2^{-n}).$$

In what follows fix some

$$m > 3/\alpha, \quad (5.1)$$

and  $\theta \in (0, 1)$  arbitrary small. Define

$$O_n := \{\omega : \text{there exists } k \in [0, 2^n - 1] \text{ such that}$$

$$\sup_{s \in (t - 2^{-\alpha n} n^{\alpha^2 m/3}, t)} X_s(I_k^{(n)}) \geq 2^{-n} n^{2m\alpha/3}\}$$

and

$$B_n = B_n(\theta) := \{\omega : \text{there exists } k \in [0, 2^n - 1] \text{ such that}$$

$$I_k^{(n)} \cap \{x : X_t(x) \geq \theta\} \neq \emptyset$$

$$\text{and } \inf_{s \in (t - 2^{-\alpha n} n^{-\alpha m}, t)} X_s(I_k^{(n)}) \leq 2^{-n} n^{-2m}\}.$$

**Lemma 5.3.** *There exists a constant  $C$  such that*

$$\mathbf{P}(O_n) \leq C n^{-m\alpha/3}, \quad n \geq 1.$$

The proof is an almost word-for-word repetition of the proof of Lemma 5.5 in [4], and we omit it.

**Lemma 5.4.** *There exists a constant  $C$  such that*

$$\mathbf{P}(B_n \cap A^\varepsilon) \leq C \theta^{-1} n^{-\alpha m/3}, \quad n \geq 1.$$

*Proof.* Define

$$\tau_n := \inf \left\{ s \in (t - 2^{-\alpha n} n^{-\alpha m}, t) : X_s(I_k^{(n)}) \leq 2^{-n} n^{-2m} \text{ for some } k \in [0, 2^n - 1] \right\}.$$

If  $\omega \in B_n$ , then it is easy to see that there exists  $\tilde{k}_n$  and a sequence  $\{s_j\}_{j=1}^\infty$  such that  $s_j \downarrow \tau_n$ , as  $j \rightarrow \infty$ , and

$$\lim_{j \rightarrow \infty} X_{s_j}(I_{\tilde{k}_n}^{(n)}) \leq 2^{-n} n^{-2m}.$$

By the right continuity of the measure valued process  $\{X_t\}_{t \geq 0}$  we get that

$$X_{\tau_n}(I_{\tilde{k}_n}^{(n)} \setminus \{\tilde{k}_n 2^{-n}\}) \leq 2^{-n} n^{-2m}.$$

Since  $X$  has only positive jumps in the form of atomic measures and these jumps do not occur at the fixed points of space, we immediately deduce that, in fact,

$$X_{\tau_n}(I_{\tilde{k}_n}^{(n)}) \leq 2^{-n} n^{-2m}.$$

Put

$$\tilde{B}_n := \left[ \frac{\tilde{k}_n}{2^n} + \frac{1}{2^{n+1}} - 2^{-n}n^{-m}, \frac{\tilde{k}_n}{2^n} + \frac{1}{2^{n+1}} + 2^{-n}n^{-m} \right].$$

If

$$\tilde{E}^{(n)} := \left\{ \omega : I_{\tilde{k}_n}^{(n)} \cap \{x : X_t(x) > \theta\} \neq \emptyset \right\},$$

then, according to the Hölder continuity with the minimal exponent  $\eta_c$ , we have on the event  $A^\varepsilon \cap \tilde{E}^{(n)}$ , that for all  $n$  sufficiently large,  $X_t(x) \geq \theta/2$  on the whole set  $\tilde{B}_n$ . Thus,

$$\begin{aligned} \theta 2^{-n} n^{-m} \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon) &= \frac{\theta}{2} |\tilde{B}_n| \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon) \\ &\leq \mathbf{E} \left[ X_t(\tilde{B}_n) 1\{\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon\} \right] \\ &\leq \mathbf{E} \left[ X_t(\tilde{B}_n) 1\{\tau_n < t, \tilde{O}_n^c\} \right]. \end{aligned} \quad (5.2)$$

where

$$\tilde{O}_n^c := \{X_{\tau_n}(I_k^{(n)}) \leq 2^{-n}n^{2m\alpha/3}, \text{ for all } k = 0, \dots, 2^n - 1\},$$

and the last inequality in (5.2) follows since  $O_n^c \subset \tilde{O}_n^c$ .

Using the strong Markov property, we then obtain

$$\theta 2^{-n} n^{-m} \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon) \leq \mathbf{E} \left[ S_{t-\tau_n} X_{\tau_n}(\tilde{B}_n) 1\{\tau_n < t, \tilde{O}_n^c\} \right] e^{|a|t}. \quad (5.3)$$

It is clear that

$$\mathbf{E} \left[ S_{t-\tau_n} X_{\tau_n}(\tilde{B}_n) 1\{\tau_n < t, \tilde{O}_n^c\} \right] \quad (5.4)$$

$$= \mathbf{E} \left[ \int_{\mathbb{R}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z) dy 1\{\tau_n < t, \tilde{O}_n^c\} \right]. \quad (5.5)$$

Since  $X_{\tau_n}(I_{\tilde{k}_n}^{(n)}) \leq 2^{-n}n^{-2m}$  on the event  $\{\tau_n < t\}$ , we have

$$\mathbf{E} \left[ \int_{I_{\tilde{k}_n}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z) dy 1\{\tau_n < t, \tilde{O}_n^c\} \right] \leq 2^{-n}n^{-2m}. \quad (5.6)$$

Recalling that  $\tau_n > t - 2^{-\alpha n}n^{-\alpha m}$  and using the scaling property of the kernel  $p^\alpha$  together with the bound (4.11) we get

$$\begin{aligned} p_{t-\tau_n}^\alpha(y-z) &= (t-\tau_n)^{-1/\alpha} p_1^\alpha \left( \frac{y-z}{(t-\tau_n)^{1/\alpha}} \right) \\ &\leq C(t-\tau_n)|y-z|^{-\alpha-1} \\ &\leq C2^{-\alpha n}n^{-\alpha m}|y-z|^{-\alpha-1}. \end{aligned}$$

Further, if  $z \in I_{\tilde{k}_n \pm j}^{(n)}$  and  $y \in \tilde{B}_n$ , then

$$\begin{aligned} |y-z| &\geq (j-1)2^{-n} + (1-n^{-m})2^{-n} = (j-n^{-m})2^{-n} \\ &\geq \frac{1}{2}j2^{-n}, \quad \forall n \geq 2, j \geq 1. \end{aligned}$$

Combining the last two bounds, we get

$$\begin{aligned} \int_{I_{\tilde{k}_n \pm j}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z)dy &\leq \int_{I_{\tilde{k}_n \pm j}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} Cj^{-\alpha-1}2^{(\alpha+1)n}2^{-\alpha n}n^{-\alpha m}dy \\ &= Cj^{-\alpha-1}n^{-(\alpha+1)m}X_{\tau_n}(I_{\tilde{k}_n \pm j}^{(n)}), \quad \forall n \geq 2, j \geq 1. \end{aligned}$$

On the event  $\{\tau_n < t\} \cap \tilde{O}_n^c$  we then have

$$\begin{aligned} \int_{I_{\tilde{k}_n \pm j}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z)dy &\leq Cj^{-\alpha-1}2^{-n}n^{-(\alpha+1)m+2m\alpha/3} \\ &= Cj^{-\alpha-1}2^{-n}n^{-(\frac{1}{3}\alpha+1)m}, \quad \forall n \geq 2, j \geq 1. \end{aligned} \tag{5.7}$$

Consequently, by summing up (5.7) over  $j \geq 1$ , we get

$$\mathbf{E} \left[ \int_{\mathbb{R} \setminus I_{\tilde{k}_n}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z)dy 1_{\{\tau_n < t, \tilde{O}_n^c\}} \right] \leq C2^{-n}n^{-(\frac{1}{3}\alpha+1)m}, \quad n \geq 2. \tag{5.8}$$

This and (5.6) imply that (5.4) is bounded by

$$C2^{-n}(n^{-(\frac{1}{3}\alpha+1)m} + n^{-2m}), \quad n \geq 2. \tag{5.9}$$

Combining (5.3), (5.9), and using the trivial bound for  $n = 1$ , we obtain

$$\theta \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon) \leq Cn^{-\frac{1}{3}\alpha m} + n^{-m} \leq Cn^{-\alpha m/3}, \quad n \geq 1.$$

In view of Lemma 5.3,

$$\mathbf{P}(\tau_n < t, \tilde{E}^{(n)}, A^\varepsilon) \leq \mathbf{P}(O_n) + \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon) \leq C\theta^{-1}n^{-\alpha m/3}, \quad n \geq 1.$$

This completes the proof of the lemma.  $\square$

Set

$$q := \frac{(\alpha+3)m}{(\beta+1)(\eta-\eta_c)}$$

and define

$$\begin{aligned} A_k^{(n)} &:= \left\{ \Delta X_s(I_{k-2n^q-2}^{(n)}) \geq 2^{-(\eta+1)n} \right. \\ &\quad \left. \text{for some } s \in [t - 2^{-\alpha n}n^{-\alpha m}, t - 2^{-\alpha(n+1)}(n+1)^{-\alpha m}] \right\}. \end{aligned}$$

Define also

$$J_k^{(n)} := \left[ \frac{k}{2^n} - 2^{-(\beta+1)(\eta-\eta_c)n}n^{(\alpha+3)m}, \frac{k+1}{2^n} + 2^{-(\beta+1)(\eta-\eta_c)n}n^{(\alpha+3)m} \right)$$

and

$$\tilde{J}_\eta := \limsup_{n \rightarrow \infty} \bigcup_{k=2n^q+2}^{2^n-1} J_k^{(n)} 1_{\{A_k^{(n)}\}}.$$

**Lemma 5.5.** *On the event  $A^\varepsilon$ ,*

$$\{X_t(x) \geq \theta, x \in (0, 1)\} \subseteq \tilde{J}_\eta, \quad \mathbf{P} - \text{a.s.}$$

*Proof.* We estimate the probability of the event  $E_n \cap A^\varepsilon$ , where

$$E_n := \left\{ \omega : \{X_t(x) \geq \theta, x \in (0, 1)\} \subseteq \bigcup_{k=2n^q+2}^{2^n-1} J_k^{(n)} 1\{A_k^{(n)}\} \right\}.$$

It follows from Lemma 5.4 that

$$\begin{aligned} \mathbf{P}(E_n^c \cap A^\varepsilon) &\leq \mathbf{P}(E_n^c \cap B_n \cap A^\varepsilon) + \mathbf{P}(E_n^c \cap B_n^c \cap A^\varepsilon) \\ &\leq C\theta^{-1}n^{-\alpha m/3} + \mathbf{P}(E_n^c \cap B_n^c \cap A^\varepsilon). \end{aligned} \quad (5.10)$$

For any  $k = 0, \dots, 2^n - 1$ , the compensator measure of  $\mathcal{N}(dr, dy, ds)$  (the jump measure for  $X$  — see discussion after (2.3)), on

$$\begin{aligned} \mathcal{J}_1^{(n)} \times I_k^{(n)} \times \mathcal{J}_2^{(n)} \\ \equiv [2^{-(\eta+1)n}, \infty) \times I_k^{(n)} \times [t - 2^{-\alpha n}n^{-\alpha m}, t - 2^{-\alpha(n+1)}(n+1)^{-\alpha m}], \end{aligned}$$

is given by the formula

$$1\{(r, y, s) \in \mathcal{J}_1^{(n)} \times I_k^{(n)} \times \mathcal{J}_2^{(n)}\} \varrho r^{-2-\beta} dr X_s(dy) ds.$$

If

$$k \in K_\theta \equiv \{l : I_l^{(n)} \cap \{x \in (0, 1) : X_t(x) \geq \theta\} \neq \emptyset\},$$

then, by the definition of  $B_n$ , we have  $X_s(I_k^{(n)}) \geq 2^{-n}n^{-2m}$ , for  $s \in \mathcal{J}_2^{(n)}$ , on the event  $A^\varepsilon \cap B_n^c$ . Therefore, on  $A^\varepsilon \cap B_n^c$ , and on the set

$$\mathcal{J}_1^{(n)} \times \{y \in (0, 1) : X_t(y) \geq \theta\} \times \mathcal{J}_2^{(n)}$$

the compensator measure  $\hat{\mathcal{N}}(dr, dy, ds)$  of  $\mathcal{N}(dr, dy, ds)$  is bounded from below by

$$\hat{\Gamma}(dr, dy, ds) \equiv \varrho r^{-2-\beta} dr n^{-2m} dy ds. \quad (5.11)$$

By standard argument it is easy to construct the Poisson point process  $\Gamma(dr, dx, ds)$  on  $\mathbf{R}_+ \times (0, 1) \times \mathbf{R}_+$  with intensity measure  $\hat{\Gamma}$  given by (5.11) on the whole space  $\mathbf{R}_+ \times (0, 1) \times \mathbf{R}_+$  such that on  $A^\varepsilon \cap B_n^c$ ,

$$\Gamma(dr, dy, ds) \leq \mathcal{N}(dr, dy, ds), \quad \text{for } (r, y, s) \in \mathcal{J}_1^{(n)} \times \{y \in (0, 1) : X_t(y) \geq \theta\} \times \mathcal{J}_2^{(n)}.$$

Now, define

$$\xi_k^{(n)} = 1 \left\{ \Gamma \left( \mathcal{J}_1^{(n)} \times I_{k-2n^q-2}^{(n)} \times \mathcal{J}_2^{(n)} \right) \geq 1 \right\}, \quad k \geq 2n^q + 2.$$

Clearly on  $A^\varepsilon \cap B_n^c$  and for  $k$  such that  $k - 2n^q - 2 \in K_\theta$ ,

$$\xi_k^{(n)} \leq 1\{A_k^{(n)}\}.$$

Moreover, by construction  $\{\xi_k^{(n)}\}_{k=2n^q+2}^{2^n+2n^q+1}$  is a collection of independent identically distributed Bernoulli random variables with success probabilities

$$\begin{aligned} p^{(n)} &\equiv \hat{\Gamma} \left( \mathcal{J}_1^{(n)} \times I_{k-2n^q-2}^{(n)} \times \mathcal{J}_2^{(n)} \right) \\ &= C 2^{(\eta-\eta_c)(1+\beta)n-n} n^{-(\alpha+2)m}. \end{aligned}$$

From the above coupling with the Poisson point process  $\Gamma$ , it is easy to see that

$$\mathbf{P}(E_n^c \cap B_n^c \cap A^\varepsilon) \leq \mathbf{P}(\tilde{E}_n^c), \quad (5.12)$$

where

$$\tilde{E}_n = \left\{ (0, 1) \subseteq \bigcup_{k=2n^q+2}^{2^n+2n^q+1} J_k^{(n)} \xi_k^{(n)} \right\}.$$

Let  $L(n)$  be the length of the longest run of zeros in the sequence  $\{\xi_k^{(n)}\}_{k=2n^q+2}^{2^n+2n^q+1}$ . Clearly,

$$\mathbf{P}(\tilde{E}_n^c) \leq \mathbf{P}(L^{(n)} \geq 2^{n-(\beta+1)(\eta-\eta_c)n} n^{m(\alpha+3)}).$$

and it is also obvious that

$$\mathbf{P}(L^{(n)} \geq j) \leq 2^n p^{(n)} (1 - p^{(n)})^j, \forall j \geq 1.$$

Use this with the fact that, by (5.1),  $m > 1$ , to get that

$$\mathbf{P}(\tilde{E}_n^c) \leq \exp \left\{ -\frac{1}{2} n^m \right\} \quad (5.13)$$

for all  $n$  sufficiently large. to see that the sequence  $\mathbf{P}(E_n^c \cap A^\varepsilon)$  is summable. Applying Borel-Cantelli, we complete the proof of the lemma.  $\square$

Set

$$V_k^{(n)} := \left[ \frac{k}{2^n} - 2^{-n} n^q, \frac{k+1}{2^n} + 2^{-n} n^q \right),$$

and

$$\tilde{V}_\eta := \limsup_{n \rightarrow \infty} \bigcup_{k=2n^q+2}^{2^n-1} V_k^{(n)} \mathbf{1}\{A_k^{(n)}\}.$$

Define

$$h_\eta(x) := x^{(\beta+1)(\eta-\eta_c)} \log^2 \frac{1}{x}$$

and

$$\mathcal{H}_\eta(A) := \liminf_{\epsilon \rightarrow 0} \left\{ \sum_{j=1}^{\infty} h_\eta(|I_j|), A \in \bigcup_{j=1}^{\infty} I_j \text{ and } |I_j| \leq \epsilon \right\}.$$

Combining Lemma 5.5 and Theorem 2 from [7], one can easily get

**Corollary 5.6.** *On the event  $A^\varepsilon$ ,*

$$\mathcal{H}_\eta(\tilde{V}_\eta) > 0, \quad \mathbf{P} - \text{a.s.}$$

*and, consequently, on  $A^\varepsilon$ ,  $\dim(\tilde{V}_\eta) \geq (\beta+1)(\eta-\eta_c)$ ,  $\mathbf{P}$ -a.s.*

Fix some integer  $Q > 1$ . According to Lemma 2.15 from [4] (see also (4.21), (4.22)) there exist spectrally positive  $(1+\beta)$ -stable processes  $L_{n,l,r}^\pm$  such that

$$\begin{aligned} \tilde{Z}_s^{2,\eta}(l2^{-Qn}, r2^{-Qn}) &= L_{n,l,r}^+(T_+^{n,l,r}(s)) - L_{n,l,r}^-(T_-^{n,l,r}(s)), \\ 0 \leq l < r \leq 2^{Qn}, \quad 0 \leq s \leq t, \end{aligned} \quad (5.14)$$

where

$$T_\pm^{n,l,r}(s) = \int_0^s du \int_{\mathbb{R}} X_u(dy) \left( (\tilde{p}_{t-u}^{\alpha,\eta}(l2^{-Qn} - y, r2^{-Qn} - y))^\pm \right)^{1+\beta}, \quad 0 \leq s \leq t.$$

We need to introduce more notation related to the event  $A_k^{(n)}$ . If  $A_k^{(n)}$  occurs then there is a jump of the process  $X$  at time  $s_k^n$  such that

$$\Delta X_{s_k^n}(I_{k-2n^q-2}^{(n)}) \geq 2^{-(\eta+1)n}, \quad (5.15)$$

and

$$s_k^n \in [t - 2^{-\alpha n} n^{-\alpha m}, t - 2^{-\alpha(n+1)}(n+1)^{-\alpha m}). \quad (5.16)$$

Let  $y_k^n$  denote the spatial position of that jump. Now put

$$l_k^n = \lfloor 2^{Qn} y_k^n \rfloor,$$

and for every  $x \in (0, 1)$  define

$$\tilde{k}_n(x) = \lfloor 2^{Qn} x \rfloor.$$

To simplify notation, in what follows, for any  $n, l, r$ , we denote by  $\Delta L_{n,l,r}^+$  the maximal jump of  $L_{n,l,r}^+$ , that is,

$$\Delta L_{n,l,r}^+ := \sup_{s \leq t} L_{n,l,r}^+(T_+^{n,l,r}(s)).$$

Also set

$$\mathbf{L}_{n,l,r}^\pm := L_{n,l,r}^\pm(T_\pm^{n,l,r}(t)). \quad (5.17)$$

**Lemma 5.7.** *Let  $\eta \in (\eta_c, \bar{\eta}_c)$ , and fix arbitrary integer  $R > 0$ . There exist constants  $C_{(5.7)}$  and  $N_{(5.7)}$  sufficiently large, such that for all  $n \geq N_{(5.7)}$  and all  $r_k^n \in \{\tilde{k}_n(x) - R, \tilde{k}_n(x) - R + 1, \dots, \tilde{k}_n(x)\}$ ,*

$$A_k^{(n)} \subset \left\{ \Delta L_{n,l_k^n,r_k^n}^+ \geq C_{(5.7)} 2^{-\eta n} n^m, \forall x \in V_k^{(n)} \right\}, \quad k = 2n^q + 2, \dots, 2^n.$$

*Proof.* Fix  $n$  sufficiently large (to be chosen later) and  $k \in \{2n^q + 2, \dots, 2^n\}$ . In what follows we assume that  $A_k^{(n)}$  occurs. Then we have to show that

$$\Delta L_{n,l_k^n,r_k^n}^+ \geq C_{(5.7)} 2^{-\eta n} n^m, \quad \forall x \in V_k^{(n)}. \quad (5.18)$$

Fix arbitrary  $x \in V_k^{(n)}$ . Recall that  $(s_k^n, y_k^n)$  denote a space-time location of a jump of  $X$  that appears in the definition of  $A_k^{(n)}$ . To simplify notation, to the end of the lemma, we will suppress the superindex  $n$  in  $l_k^n, r_k^n, y_k^n$ . If  $A_k^{(n)}$  occurred then

$$\Delta L_{n,l_k,r_k}^+ \geq 2^{-(\eta+1)n} (\tilde{p}_{t-s_k}^{\alpha,\eta} (l_k 2^{-Qn} - y_k, r_k 2^{-Qn} - y_k))_+. \quad (5.19)$$

So to verify the lemma, we have to obtain suitable strictly positive lower bound for  $\tilde{p}_{t-s_k}^{\alpha,\eta} (l_k 2^{-Qn} - y_k, r_k 2^{-Qn} - y_k)$ .

First, we will obtain a lower bound for  $p_{t-s_k}^\alpha (l_k 2^{-Qn} - y_k)$ :

$$\begin{aligned} p_{t-s_k}^\alpha (l_k 2^{-Qn} - y_k) &= (t - s_k)^{-1/\alpha} p_1 \left( (t - s_k)^{-1/\alpha} (l_k 2^{-Qn} - y_k) \right) \\ &\geq 2^n n^m p_1 \left( (t - s_k)^{-1/\alpha} (l_k 2^{-Qn} - y_k) \right), \end{aligned} \quad (5.20)$$

where the last inequality follows by (5.16). By definition of  $l_k$  we get

$$|l_k 2^{-Qn} - y_k| \leq 2^{-Qn}, \quad (5.21)$$

and this with again (5.16) and monotonicity of  $p_1(\cdot)$  implies

$$p_1 \left( (t - s_k)^{-1/\alpha} (l_k 2^{-Qn} - y_k) \right) \geq p_1 \left( 2^{-(Q-1)n+1} (n+1)^m \right) \geq p_1(1) \quad (5.22)$$

for all  $n$  sufficiently large. Let  $N_{(5.22)}$  be sufficiently large such that (5.22) holds for all  $n \geq N_{(5.22)}$ . Then (5.20), (5.22) imply

$$\begin{aligned} p_{t-s_k}^\alpha(l_k 2^{-Qn} - y_k) &\geq 2^n p_1(1) n^m \\ &= C_{(5.23)} 2^n n^m, \quad \forall n \geq N_{(5.22)}. \end{aligned} \quad (5.23)$$

Next, by definition of  $A_k^{(n)}, V_k^{(n)}$ , we easily get, for all sufficiently large  $n$ ,

$$-y_k + r_k 2^{-Qn} \geq 2^{-n}(2n^q + 1) - (2^{-n}n^q + R2^{-Qn}) \geq 2^{-n-1}n^q. \quad (5.24)$$

Use this, the bound on  $t - s_k$ , and to get

$$\begin{aligned} p_{t-s_k}^\alpha(r_k 2^{-Qn} - y_k) &= (t - s_k)^{-1/\alpha} p_1 \left( (t - s_k)^{-1/\alpha} (r_k 2^{-Qn} - y_k) \right) \\ &\leq C_{(4.11)} (t - s_k)^{-1/\alpha} \left( (t - s_k)^{-1/\alpha} (r_k 2^{-Qn} - y_k) \right)^{-\alpha-1} \\ &= C_{(4.11)} (t - s_k) (r_k 2^{-Qn} - y_k)^{-\alpha-1} \\ &\leq C_{(4.11)} 2^{-\alpha n} n^{-\alpha m} 2^{(n+1)(\alpha+1)} n^{-q(\alpha+1)} \\ &= C_{(5.25)} 2^n n^{-\alpha m - q(\alpha+1)}. \end{aligned} \quad (5.25)$$

Next we will bound from above the quantity

$$\left| (l_k 2^{-Qn} - r_k 2^{-Qn}) p_{t-s_k}^{\alpha, \prime} (r_k 2^{-Qn} - y_k) \right|,$$

where  $p_t^{\alpha, \prime}(z) \equiv \frac{\partial p^\alpha(z)}{\partial z}$ . It is easy to check that

$$\begin{aligned} \left| p_{t-s_k}^{\alpha, \prime} (r_k 2^{-Qn} - y_k) \right| &\leq C (t - s_k)^{-2/\alpha} p_1^\alpha((t - s_k)^{-1/\alpha} (r_k 2^{-Qn} - y_k)/2) \\ &= C (t - s_k)^{-1/\alpha} p_{t-s_k}^\alpha((r_k 2^{-Qn} - y_k)/2) \\ &\leq C_{(5.26)} 2^n n^m 2^n n^{-\alpha m - q(\alpha+1)}, \end{aligned} \quad (5.26)$$

where the last inequality follows by (5.25) and the bound on  $t - s_k$ . Since

$$|l_k 2^{-Qn} - r_k 2^{-Qn}| \leq 3 \cdot 2^{-n} n^q,$$

this implies

$$\left| (l_k 2^{-Qn} - r_k 2^{-Qn}) p_{t-s_k}^{\alpha, \prime} (r_k 2^{-Qn} - y_k) \right| \leq 3C_{(5.26)} 2^n n^{-m(\alpha-1)-q\alpha}. \quad (5.27)$$

Then by definition of  $\tilde{p}_t^{\alpha, \eta}$ , (5.23), (5.25), (5.27) we immediately get, that there exists  $N_{(5.28)} \geq N_{(5.23)}$ , such that, for any  $\eta \in (\eta_c, \bar{\eta}_c)$ ,

$$\begin{aligned} \tilde{p}_{t-s_k}^{\alpha, \eta} (l_k 2^{-Qn} - y_k, r_k 2^{-Qn} - y_k) &\geq C_{(5.23)} 2^n n^m - C_{(5.25)} 2^n n^{-m\alpha - q(\alpha+1)} - 3C_{(5.26)} 2^n n^{-m(\alpha-1)-q\alpha} \\ &\geq \frac{1}{2} C_{(5.23)} 2^n n^m, \quad \forall n \geq N_{(5.28)}. \end{aligned} \quad (5.28)$$

Substitute the above lower bound into (5.19) and the result follows immediately.  $\square$

**Lemma 5.8.** *Let  $\eta \in (\eta_c, \bar{\eta}_c)$ , and fix arbitrary integer  $R > 0$ . On  $A^\varepsilon$ , for every  $x \in \tilde{V}_\eta$  there exists a (random) sequence  $\{(n_j, k_j)\}$ , such that*

$$\mathbf{L}_{n_j, l_{k_j}^{n_j}, r_{k_j}^{n_j}}^+ \geq C 2^{-\eta n_j} n_j^m$$

for all  $r_{k_j}^{n_j} \in [\tilde{k}_{n_j}(x) - R, \tilde{k}_{n_j}(x) - R + 1, \dots, \tilde{k}_{n_j}(x)]$ .



*Proof.* Recall (4.24) to get that on  $A^\varepsilon$ , and any  $l, r$ ,

$$T_+^{n,l,r}(t) \leq \hat{T}^{n,l,r}(l2^{-Qn}, r2^{-Qn}) = C_{(4.24)}(|r-l|2^{-Qn})^{\alpha-\beta-\varepsilon_1}, \quad (5.29)$$

and take  $\varepsilon_1 < (\bar{\eta}_c - \eta)(\beta + 1)/2$ . This, Lemma 5.7 and Lemma 2.4 from [4], imply that for all  $n$  sufficiently large, and, for any  $k \in [2n^q + 2, 2^n - 1]$ ,

$$\begin{aligned} & \mathbf{P} \left( A^\varepsilon \cap A_k^{(n)} \cap \{ \mathbf{L}_{n,l_k^n, r_k^n}^+ \leq C_{5.7} 2^{-\eta n-1} n^m \} \right) \\ & \leq \mathbf{P} \left( A^\varepsilon \cap \{ \Delta L_{n,l_k^n, r_k^n}^+ \geq C_{5.7} 2^{-\eta n} n^m \} \cap \{ \mathbf{L}_{n,l_k^n, r_k^n}^+ \leq C_{5.7} 2^{-\eta n-1} n^m \} \right) \\ & \leq \sum_{\substack{l : 2^{-Qn}l \in I_k^{(n)}, \\ r : 2^{-Qn}r \in V_k^{(n)}}} \mathbf{P} \left( A^\varepsilon \cap \{ \Delta L_{n,l,r}^+ \geq C_{5.7} 2^{-\eta n} n^m \} \cap \{ \mathbf{L}_{n,l,r}^+ \leq C_{5.7} 2^{-\eta n-1} n^m \} \right) \\ & \leq \sum_{\substack{l : 2^{-Qn}l \in I_k^{(n)}, \\ r : 2^{-Qn}r \in V_k^{(n)}}} \mathbf{P} \left( \inf_{s \leq \hat{T}^{n,l,r}(l2^{-Qn}, r2^{-Qn})} L_{n,l,r}^+(s) \leq -C_{5.7} 2^{-\eta n-1} n^m \right) \\ & \leq 2^{(Q-1)n} 5n^q \exp \left\{ -c 2^{(\bar{\eta}_c - \eta)(\beta+1)n/(2\beta)} \right\}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \mathbf{P} \left( A^\varepsilon \cap A_k^{(n)} \cap \{ \mathbf{L}_{n,l_k^n, r_k^n}^+ \leq C_{5.7} 2^{-\eta n-1} n^m \} \right) \\ & \leq 2^{(2Q-1)n} 5n^q \exp \left\{ -c 2^{(\bar{\eta}_c - \eta)(\beta+1)n/(2\beta)} \right\} \\ & \leq \exp \left\{ -2^{(\bar{\eta}_c - \eta)(\beta+1)n/(4\beta)} \right\}, \end{aligned}$$

for all  $n$  sufficiently large. Using Borel-Cantelli, we get that with probability one

$$\bigcup_{k=2n^q+2}^{2^n-1} \{ A^\varepsilon \cap A_k^{(n)} \cap \{ \mathbf{L}_{n,l_k^n, r_k^n}^+ \leq C_{5.7} 2^{-\eta n-1} n^m \} \}$$

occurs only finite number of times. Let  $x \in \tilde{V}_\eta$  be arbitrary. By definition of  $\tilde{V}_\eta$ , there exists a (random) sequence  $\{(n_j, k_j)\}$  such that

$$x \in V_{k_j}^{(n_j)}, \text{ and } 1_{A_{k_j}^{(n_j)}} = 1, \forall j \geq 1.$$

Therefore, on the set  $A^\varepsilon$  we have

$$\mathbf{L}_{n,l_{k_j}^{n_j}, r_{k_j}^{n_j}}^+ > C_{(5.7)} 2^{-\eta n_j-1} n_j^m,$$

for all  $j$  sufficiently large and all  $r_{n_j}^{k_j} \in [\tilde{k}_{n_j}(x) - R, \tilde{k}_{n_j}(x)]$ .  $\square$

If we recall (5.14), the above lemma implies, that it is may be possible to destroy the Hölder continuity, of any index greater than  $\eta$ , of the process on the set  $\tilde{V}_\eta$ . For this purpose we use processes  $L_{n,k,l}^+$ . However, as again one can see from (5.14), to finish the proof one should show loosely speaking that on "significant" part of  $\tilde{V}_\eta$  there is no compensation of "big" values of  $L^+$  by "big" values of  $L^-$ . This is the most difficult part of the argument and the rest of this section is devoted to its proof.

First, fix arbitrary positive constants  $\rho, c, \nu$  such that

$$\rho < 10^{-2}\gamma, \quad \nu \in \left( \frac{\alpha\gamma + 5\rho}{\eta_c}, 10^{-1} \right), \quad c \in \left( \frac{10}{2-\eta}, \frac{1}{10\rho} \right) \quad (5.30)$$

Define

$$\begin{aligned} G_k^{(n)} := & \left\{ \text{there exist at least two jumps of } M, \text{ of the form } r\delta_{(s,y)}, \right. \\ & \text{satisfying } r \geq 2^{-(\eta+1+2\rho+2c\rho)n}, s \in [t - 2^{-\alpha(1-c\rho)n}, t - 2^{-\alpha(1+c\rho)n}) \\ & \left. \text{and } y \in \left[ \frac{k}{2^n} - 2^{-n(1-c\rho)(1-\nu)}, \frac{k+1}{2^n} + 2^{-n(1-c\rho)(1-\nu)} \right] \right\}. \end{aligned} \quad (5.31)$$

and

$$\tilde{G}_\eta := \limsup_{n \rightarrow \infty} \bigcup_{k=0}^{2^n-1} I_k^{(n)} 1\{G_k^{(n)}\}.$$

Informally,  $\tilde{G}_\eta$  is the set in certain proximity of which there are at least two "big" jumps of  $M$ . If one of the jumps, appears in  $L^+$ , and another in  $L^-$ , they may compensate each other; however in the next lemma we will show that the Hausdorff dimension of  $\tilde{G}_{eta}$  is small.

**Lemma 5.9.** *On  $A^\varepsilon$ ,*

$$\dim(\tilde{G}_\eta) \leq (2(\beta+1)(\eta-\eta_c)-1)^+ + 2(\nu+8(c+1)\rho), \quad \mathbf{P} - \text{a.s.}$$

*Proof.* On the event  $O_n^c$  we have the following upper bound for the intensity of the jumps in  $G_k^{(n)}$ :

$$\begin{aligned} & \int_{t-2^{-\alpha(1-c\rho)n}}^{t-2^{-\alpha(1+c\rho)n}} ds X_s \left( \left[ \frac{k}{2^n} - 2^{-n(1-c\rho)(1-\nu)}, \frac{k+1}{2^n} + 2^{-n(1-c\rho)(1-\nu)} \right] \right) \\ & \quad \times \int_{2^{-(\eta+1+2\rho+2c\rho)n}} q r^{-2-\beta} dr \\ & \leq C 2^{n-n(1-c\rho)(1-\nu)} 2^{-n} n^{2m} 2^{-\alpha(1-\rho c)n} 2^{(\beta+1)(\eta+1+2\rho+2c\rho)} \\ & \leq C 2^{-n+n(\beta+1)(\eta-\eta_c)+\delta n}, \end{aligned}$$

where  $\delta = \nu + 7(c+1)\rho$ . Since the number of such jumps can be represented by means of a time-changed standard Poisson process, the probability to have at least two such jumps is bounded by the square of the above bound, i.e.,

$$\mathbf{P}(O_n^c \cap G_k^{(n)}) \leq C 2^{-2n+2n(\beta+1)(\eta-\eta_c)+2\delta n} =: p^{(n)}.$$

Combining this bound with Lemma 5.3 and the Markov inequality, we get

$$\begin{aligned} \mathbf{P} \left( \sum_{k=0}^{2^n-1} 1\{G_k^{(n)}\} \geq 2^{n+\varepsilon n} p^{(n)} \right) & \leq \mathbf{P}(O_n) + \mathbf{P} \left( \sum_{k=0}^{2^n-1} 1\{G_k^{(n)}\} \geq 2^{n+\varepsilon n} p^{(n)}; O_n^c \right) \\ & \leq C n^{-m\alpha/3} + \frac{2^n \mathbf{P}(G_1^{(n)})}{2^{n+\varepsilon n} p^{(n)}} \\ & \leq C n^{-m\alpha/3} + 2^{-\varepsilon n}. \end{aligned}$$

If  $2(\beta+1)(\eta-\eta_c)+2\delta < 1$ , then, choosing  $\varepsilon$  sufficiently small, we obtain

$$\mathbf{P} \left( \sum_{k=0}^{2^n-1} 1\{G_k^{(n)}\} \geq 1 \right) \leq C 2^{-mn} + 2^{-\varepsilon n}.$$

Applying finally Borel-Cantelli, we conclude that  $\tilde{G}_\eta = \emptyset$  almost surely. In particular,  $\dim(\tilde{G}_\eta) = 0$  with probability one.

Assume now that  $2(\beta + 1)(\eta - \eta_c) + 2\delta \geq 1$ . Applying Borel-Cantelli once again, we see that the number of indices  $k$  with  $1\{G_k^{(n)}\} = 1$  is bounded by  $2^{n+\varepsilon n} p^{(n)}$ . Noting that  $\tilde{G}_\eta$  can be covered by  $\bigcup_{k=1}^{2^n} I_k^{(n)} 1\{G_k^{(n)}\}$  and

$$\sum_{n=1}^{\infty} 2^{n+\varepsilon n} p^{(n)} 2^{-\theta n} < \infty \quad \text{for all } \theta > 2(\beta + 1)(\eta - \eta_c) + 2\delta - 1 + \varepsilon,$$

we infer that

$$\dim(\tilde{G}_\eta) \leq 2(\beta + 1)(\eta - \eta_c) + 2\delta - 1 + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , we get the desired result.  $\square$

Define

$$F_k^{(N)} := \left\{ \Delta X_s(y) \geq (t-s)^{\frac{1}{1+\beta}-\gamma} |y - (k+1)2^{-N}|^{\eta-\eta_c} \right. \\ \left. \text{for some } s \geq t - 2^{-\alpha N}, y \in I_k^{(N)} \right\}.$$

and

$$F^{(N)} := \bigcup_{k=0}^{2^N-1} \left( \bigcap_{j < R} F_{k+j}^{(N)} \right).$$

**Lemma 5.10.** *For every  $R > (1 - (1 + \beta)(\eta - \eta_c))^{-1}$  there exists a constant  $C = C(R)$  such that*

$$\mathbf{P}(F^{(N)}) \leq CN^{-m\alpha/3+1}.$$

*Proof.* It follows from Lemma 5.3 that

$$\mathbf{P}(\cup_{n \geq N} A_n) \leq \sum_{n=N}^{\infty} \mathbf{P}(A_n) \leq CN^{-m\alpha/3+1}.$$

Therefore

$$\begin{aligned} \mathbf{P}(F^{(N)}) &\leq \mathbf{P}\left(F^{(N)} \cap (\cap_{n \geq N} A_n^c)\right) + \mathbf{P}(\cup_{n \geq N} A_n) \\ &\leq \sum_{k=0}^{2^N-R-1} \mathbf{P}\left(\left(\bigcap_{j < R} F_{k+j}^{(N)}\right) \cap (\cap_{n \geq N} A_n^c)\right) + CN^{-m\alpha/3+1}. \end{aligned} \quad (5.32)$$

Consider a jump characterised by the triple  $(y, s, r)$ . We first assume that

$$-(t-s)^{1/\alpha} < y - (k+1)2^{-n} < 0.$$

This jump affects  $F_k^{(N)}$  if and only if  $r > (t-s)^{\frac{1}{1+\beta}-\gamma+\frac{\eta-\eta_c}{\alpha}}$ . If we consider  $s \in [t-2^{-\alpha j}, t-2^{-\alpha(j+1)})$ , then  $r$  should be greater than  $2^{-\alpha(j+1)(\frac{1}{1+\beta}-\gamma+\frac{\eta-\eta_c}{\alpha})}$ . Since

$$\sup_{s \geq t-2^{-\alpha j}} X_u([(k+1)2^{-n} - 2^{-j}, (k+1)2^{-n}]) \leq j^{2m} 2^{-j}$$

on the event  $A_j^c$ , we have the following bound for the intensity of jumps described above:

$$\begin{aligned} & \int_{t-2^{-\alpha j}}^{t-2^{-\alpha(j+1)}} du X_u([(k+1)2^{-n} - 2^{-j}, (k+1)2^{-n}]) \int_{2^{-\alpha(j+1)(\frac{1}{1+\beta}-\gamma) + \frac{\eta-\eta_c}{\alpha}}}^{\infty} \varrho r^{-2-\beta} dr \\ & \leq C j^{2m} 2^{-j(\alpha+1)} 2^{j(\alpha-\gamma\alpha(\beta+1)+(\eta-\eta_c)(\beta+1))} \\ & = C j^{2m} 2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)}. \end{aligned} \quad (5.33)$$

If  $(k+1)2^{-N} - y \in [a2^{-j}, (a+1)2^{-j}]$  with some  $a \geq 1$  then  $r$  should be bigger than  $2^{-\alpha(j+1)(\frac{1}{1+\beta}-\gamma)} a^{\eta-\eta_c} 2^{-j(\eta-\eta_c)}$ . Then, on the event  $A_j^c$ ,

$$\begin{aligned} & \int_{t-2^{-\alpha j}}^{t-2^{-\alpha(j+1)}} du X_u([a2^{-j}, (a+1)2^{-j}]) \int_{2^{-\alpha(j+1)(\frac{1}{1+\beta}-\gamma)} a^{\eta-\eta_c} 2^{-j(\eta-\eta_c)}}^{\infty} \varrho r^{-2-\beta} dr \\ & \leq C j^{2m} 2^{-j(\alpha+1)} 2^{j(\alpha-\gamma\alpha(\beta+1)+(\eta-\eta_c)(\beta+1))} a^{-(\eta-\eta_c)(\beta+1)} \\ & = C j^{2m} 2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)} a^{-(\eta-\eta_c)(\beta+1)}. \end{aligned} \quad (5.34)$$

Combining (5.33), (5.34) and noting that we can cover the interval  $I_k^{(N)}$  by the union of intervals  $[(k+1)2^{-N} - (a+1)2^{-j}, (k+1)2^{-N} - a2^{-j}]$  with  $a < 2^{j-N}$ , we see that the intensity of jumps with  $y \in I_k^{(N)}$ ,  $s \geq t - 2^{-\alpha N}$  is bounded by

$$\begin{aligned} & \sum_{j=N}^{\infty} C j^{2m} 2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)} \left( 1 + \sum_{a=1}^{2^{j-N}-1} a^{-(\eta-\eta_c)(\beta+1)} \right) \\ & \leq C \sum_{j=N}^{\infty} j^{2m} 2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)} (2^{j-N})^{1-(\eta-\eta_c)(\beta+1)} \\ & = C (2^{-N})^{1-(\eta-\eta_c)(\beta+1)} \sum_{j=N}^{\infty} j^{2m} 2^{-j\gamma\alpha(\beta+1)} \\ & \leq C (2^{-N})^{1-(\eta-\eta_c)(\beta+1)}. \end{aligned}$$

This implies that

$$\mathbf{P} \left( F_k^{(N)} \cap (\cap_{n \geq N} A_n^c) \right) \leq C (2^{-N})^{1-(\eta-\eta_c)(\beta+1)}.$$

Since the jumps can be represented by a time-changed Poisson process, we then get

$$\mathbf{P} \left( \left( \cap_{j < R} F_{k+j}^{(N)} \right) \cap (\cap_{n \geq N} A_n^c) \right) \leq C (2^{-N})^{R(1-(\eta-\eta_c)(\beta+1))}.$$

Applying this bound to summands in (5.32) we complete the proof of the lemma.  $\square$

From this lemma and the Borel-Cantelli lemma we obtain

**Corollary 5.11.** *For  $\mathbf{P}$ -a.s.  $\omega \in A^\varepsilon$  there exists  $N_{(5.11)} = N_{(5.11)}(\omega)$  such that for every  $N \geq N_{(5.11)}$  and every  $k : R \leq k < 2^N$  there exists  $j = j(k, N) \in \{1, \dots, R\}$  with  $1_{F_{k-j}^{(N)}} = 0$ .*

We need to introduce some more notation. Let

$$k_n(x) := \lfloor 2^n x \rfloor. \quad (5.35)$$

Recall  $A_{k_n(x)}^{(n)}$ , and let  $\tilde{s}$  be the time and  $\tilde{y}$  be the spatial position of a jump described in the definition of the event  $A_{k_n(x)}^{(n)}$ . Then on  $A_{k_n(x)}^{(n)}$ , fix

$$\tilde{l}_n(x) := \lfloor 2^{Qn} \tilde{y} \rfloor, \quad (5.36)$$

Moreover, since  $Q > 1$ , for every  $n \geq N_{(5.11)}$  we can define

$$\tilde{r}_n(x) = j(\tilde{k}_n(x), Qn), \quad (5.37)$$

where  $j(\cdot, \cdot)$  is defined in Corollary 5.11, and recall that  $\tilde{k}_n(x) = k_{Qn}(x)$ . We also will define three sets in  $[0, t) \times \mathbb{R}$ . For any  $x \in \mathbb{R}$ , set

$$\begin{aligned} S_{n,x}^1 &:= \{(s, y) \in [0, t) \times \mathbb{R} : |y - x| \geq (t - s)^{(1-\nu)/\alpha}\}, \\ S_{n,x}^2 &:= \{(s, y) \in ([0, t) \setminus (t - 2^{-\alpha(1-c\rho)n}, t - 2^{-\alpha(1+c\rho)n})) \times \mathbb{R} : \\ &\quad |y - x| \leq (t - s)^{(1-\nu)/\alpha}\}, \\ S_{n,x}^3 &:= \{(s, y) \in [t - 2^{-\alpha(1-c\rho)n}, t - 2^{-\alpha(1+c\rho)n}] \times \mathbb{R} : \\ &\quad |y - x| \leq (t - s)^{(1-\nu)/\alpha}, \Delta X_s(y) \leq (t - s)^{(\eta+1+3\rho)/\alpha}\}, \end{aligned}$$

and note that the last set is random. In the next lemma we will show that, under certain conditions, the jumps of  $L^-$  are small on the above sets. We will also need additional piece of notation. Let

$$S^4 := \{(s, y) \in [0, t) \times \mathbb{R} : \Delta X_s(\{y\}) > 0\}$$

be the set of points in space×time where the jumps of  $X$ , or euivalently of  $M$ , occur.

**Lemma 5.12.** *Let  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ , and  $\rho, \nu, c$  be as in (5.30). For  $\mathbf{P}$ -a.s.  $\omega \in A^\varepsilon$ , there exists  $N_{(5.12)} = N_{(5.12)}(\omega)$  such that for every  $n \geq N_{(5.12)}$  the following holds. Fix arbitrary  $x \in (0, 1) \setminus S_{\eta-2\rho}$  such that  $1\{A_{k_n(x)}^{(n)}\} = 1$ . Then there exists a constant  $C_{(5.38)} = C_{(5.38)}(\rho, \nu, c)$  such that for any  $(s, y) \in \left(\bigcup_{i=1}^3 S_{n,x}^i\right) \cap S^4$ , we have*

$$\Delta L_{n, \tilde{l}_n(x), \tilde{r}_n(x)}^-(T_{-}^{n, \tilde{l}_n(x), \tilde{r}_n(x)}(s)) \leq C_{(5.38)} 2^{-(\eta+2\rho)n}. \quad (5.38)$$

*Proof.* Fix some  $\omega \in A^\varepsilon$  and choose  $N_{(5.12)} \geq N_{(5.11)}$ ; the choice of  $N_{(5.12)}$  will be clear from the proof. Take arbitrary  $n \geq N_{(5.12)}$ . Fix also some  $x \in (0, 1) \setminus S_{\eta-2\rho}$  satisfying  $1\{A_{k_n(x)}^{(n)}\} = 1$ . Recall (5.36), (5.37), and in what follows to simplify the notation denote

$$l = \tilde{l}_n(x), \quad r = \tilde{r}_n(x).$$

It is clear that a jump, which appears in the definition of  $A_{k_n(x)}^{(n)}$ , does not produce a jump of  $L_{n,l,r}^-$ .

Recall that  $x \in (0, 1) \setminus S_{\eta-2\rho}$  means that, for any  $y \in \mathbb{R}$ ,

$$\Delta X_s(y) \leq (t - s)^{\frac{1}{\beta+1}-\gamma} |y - x|^{\eta-2\rho-\eta_c}. \quad (5.39)$$

We will treat the three regions  $S_{n,x}^i, i = 1, 2, 3$  separately.

(i) Let  $(s, y) \in S_{n,x}^1 \cap S^4$ , that is,

$$|y - x| \geq (t - s)^{(1-\nu)/\alpha}. \quad (5.40)$$

First assume  $|y - x| \leq n^q 2^{-n}$ . We will consider the cases of  $\eta < 1$  and  $\eta > 1$  separately. We start with

**Case  $\eta < 1$ .** For all  $x_1, x_2 \in \mathbb{R}$ ,

$$(p_{t-s}^{\alpha, \eta}(x_1, x_2))^- = (p_{t-s}^\alpha(x_1) - p_{t-s}^\alpha(x_2))^- \leq \frac{p_1^\alpha(0)}{(t-s)^{1/\alpha}}.$$

Combining this inequality with (5.39) and (5.40), we get

$$\begin{aligned} \Delta L_{n,l,r}^-(T_-^{n,l,r}(s)) &\leq p_1^\alpha(0)(t-s)^{\frac{1}{\beta+1}-\gamma}|y-x|^{\eta-2\rho-\eta_c}(t-s)^{-1/\alpha} \\ &= p_1^\alpha(0)(t-s)^{(\eta_c-\alpha\gamma)/\alpha}|y-x|^{\eta-2\rho-\eta_c} \\ &\leq p_1^\alpha(0)|y-x|^{\frac{\eta_c-\alpha\gamma}{1-\nu}+\eta-\eta_c-2\rho}. \end{aligned}$$

We choose  $N_{(5.12)}$  sufficiently large such that, for any  $\nu \in \left(\frac{\alpha\gamma+5\rho}{\eta_c}, 10^{-1}\right)$  and  $|y-x| \leq n^q 2^{-n}$ , (5.38) holds, for all  $n \geq N_{(5.12)}$ .

**Case  $\eta > 1$ .** Here we have

$$\begin{aligned} p_{t-s}^{\alpha, \eta}(l2^{-Qn} - y, r2^{-Qn} - y) \\ = p_{t-s}^\alpha(l2^{-Qn} - y) - p_{t-s}^\alpha(r2^{-Qn} - y) + (r-l)2^{-Qn}p_{t-s}^{\alpha, '}(r2^{-Qn} - y). \end{aligned}$$

Note that  $p_{t-s}^{\alpha, '}(z) \geq 0$  for all  $z \leq 0$ . Consequently,

$$\begin{aligned} (p_{t-s}^{\alpha, \eta}(l2^{-Qn} - y, r2^{-Qn} - y))^- &\leq (p_{t-s}^\alpha(l2^{-Qn} - y) - p_{t-s}^\alpha(r2^{-Qn} - y))^- \\ &\leq \frac{p_1^\alpha(0)}{(t-s)^{1/\alpha}} \end{aligned} \quad (5.41)$$

for all  $y \geq r2^{-Qn}$ .

In the complimentary case  $y < r2^{-Qn}$  one can easily get

$$(p_{t-s}^{\alpha, \eta}(l2^{-Qn} - y, r2^{-Qn} - y))^- \leq \frac{p_1^\alpha(0)}{(t-s)^{1/\alpha}} + |(r-l)2^{-Qn}p_{t-s}^{\alpha, '}(r2^{-Qn} - y)|. \quad (5.42)$$

If  $y \leq (r-1)2^{-Qn}$ , then  $r2^{-Qn} - y \geq (x-y)/(R+1)$ . Thus, using the bound  $p_1^{\alpha, '}(z) \leq C|z|^{-\alpha-2}$  and the scaling property, we obtain

$$\begin{aligned} -p_{t-s}^{\alpha, '}(r2^{-Qn} - y) &\leq C(t-s)^{-2/\alpha} \left( \frac{r2^{-Qn} - y}{(t-s)^{1/\alpha}} \right)^{-\alpha-2} \\ &\leq CR^{\alpha+2}(t-s)|x-y|^{-\alpha-2}. \end{aligned} \quad (5.43)$$

From this, (5.41) and (5.42) we conclude that, for all  $y$  satisfying  $|y - r2^{-Qn}| > 2^{-Qn}$ ,

$$(p_{t-s}^{\alpha, \eta}(l2^{-Qn} - y, r2^{-Qn} - y))^- \leq C \left( (t-s)^{-1/\alpha} + (r-l)2^{-Qn}(t-s)|x-y|^{-\alpha-2} \right).$$

Combining this with (5.39) we conclude that the corresponding jump  $\Delta L_{n,l,r}^-(T_-^{n,l,r}(s))$  is bounded by

$$C(t-s)^{\frac{1}{\beta+1}-\gamma}|y-x|^{\eta-2\rho-\eta_c} \left( (t-s)^{-1/\alpha} + (r-l)2^{-Qn}(t-s)|x-y|^{-\alpha-2} \right).$$

Taking into account (5.40), we see that the expression on the right hand side does not exceed

$$C \left( |y-x|^{\frac{\eta_c-\alpha\gamma}{1-\nu}+\eta-\eta_c-2\rho} + (r-l)2^{-Qn}|y-x|^{\frac{\eta-1}{1-\nu}+\nu\alpha-2\rho} \right).$$

Now it is easy to see that (5.38) remain valid for  $\eta > 1$  under additional assumption  $y \leq (r-1)2^{-Qn}$ , or  $y \geq r2^{-Qn}$ .

Now we will take care of the case  $y \in ((r-1)2^{-Qn}, r2^{-Qn})$ . By Corollary 5.11 and our definition of  $r = \tilde{r}_n(x)$  (recall again (5.37) and  $n \geq N_{(5.12)} \geq N_{(5.11)}$ ,  $Q > 1$ ) we obtain

$$\Delta X_s(y) \leq (t-s)^{\frac{1}{\beta+1}-\gamma} |y - r2^{-Qn}|^{\eta-\eta_c} \quad (5.44)$$

for all  $y \in ((r-1)2^{-Qn}, r2^{-Qn})$  and  $s \geq t - 2^{-\alpha Qn}$ . It is clear that  $y \in ((r-1)2^{-Qn}, r2^{-Qn})$  implies that  $|y-x| \leq (R+1)2^{-Qn}$ . From this and (5.40) we infer that  $(t-s) \leq (R+1)^{\alpha/(1-\nu)} 2^{-\alpha Qn/(1-\nu)}$  and, consequently,  $s \geq t - 2^{-\alpha Qn}$  for all sufficiently large  $n$ . Repeating all the arguments after (5.42) and using (5.44) insted of (5.39), we obtain (5.38).

Summarising, (5.38) is valid for all  $|y-x| \leq n^q 2^{-n}$ .

In case  $|y-x| \geq n^q 2^{-n}$  we apply Corollary 2.3 if  $\eta > 1$ , or Lemma 2.1 if  $\eta < 1$  with  $\delta = \eta + 3\rho$  (recall the bounds on  $\rho$  and  $\gamma$  to get that  $\delta < 1$  if  $\eta < 1$  and  $\delta < 2$  if  $\eta > 1$ ) to get

$$|\tilde{p}_{t-s}^{\alpha,\eta}(l2^{-Qn} - y, r2^{-Qn} - y)| \leq C \frac{(r-l)^{\delta} 2^{-Q\delta n}}{(t-s)^{(\delta+1)/\alpha}} p_1^\alpha \left( \frac{y - r2^{-Qn}}{(t-s)^{1/\alpha}} \right).$$

Since  $(r-l)2^{-Qn} \leq 4n^q 2^{-n}$  and  $r2^{-Qn} \leq x$ , we then have

$$|\tilde{p}_{t-s}^{\alpha,\eta}(l2^{-Qn} - y, r2^{-Qn} - y)| \leq C \frac{n^{q\delta} 2^{-\delta n}}{(t-s)^{(1+\delta)/\alpha}} p_1^\alpha \left( \frac{y-x}{(t-s)^{1/\alpha}} \right).$$

From this bound and (4.11), we obtain

$$\begin{aligned} \Delta L_{n,l,r}^-(T_-^{n,l,r}(s)) &\leq C(t-s)^{\frac{1}{\beta+1}-\gamma} |y-x|^{\eta-\eta_c-2\rho} \frac{n^{q\delta} 2^{-\delta n}}{(t-s)^{(\delta+1)/\alpha}} p_1^\alpha \left( \frac{y-x}{(t-s)^{1/\alpha}} \right) \\ &\leq C n^{q\delta} 2^{-\delta n} (t-s)^{\frac{1}{\beta+1}-\gamma-\frac{\delta+1}{\alpha}+\frac{\alpha+1}{\alpha}} |y-x|^{\eta-\eta_c-2\rho-\alpha-1}. \end{aligned}$$

As a result, for  $|y-x| \geq (t-s)^{(1-\nu)/\alpha}$  we have

$$\begin{aligned} \Delta L_{n,l,r}^-(T_-^{n,l,r}(s)) &\leq C n^{q\delta} 2^{-\delta n} \left( (t-s)^{1/\alpha} \right)^{\frac{\alpha}{\beta+1}-\alpha\gamma-\delta+\alpha+(\eta-\eta_c-2\rho-\alpha-1)(1-\nu)} \\ &\leq C 2^{-(\eta+2\rho)n} \left( (t-s)^{1/\alpha} \right)^{\eta_c+1-\alpha\gamma-\eta-3\rho+\alpha+(\eta-\eta_c-2\rho-\alpha-1)(1-\nu)} \\ &= C 2^{-(\eta+2\rho)n} \left( (t-s)^{1/\alpha} \right)^{\nu(\alpha+1-\eta+\eta_c+2\rho)-\alpha\gamma-5\rho}. \end{aligned}$$

Finally, recall that  $\nu \geq \frac{5\rho+\alpha\gamma}{\eta_c}$ , and then (5.38) holds with an appropriate constant  $C_{(5.38)}$ .

(ii) Let  $(s, y) \in S_{n,x}^2 \cap S^4$ .

We start with the subset of  $S_{n,x}^2$  where  $|y-x| \leq (t-s)^{(1-\nu)/\alpha}$  and  $s \leq t - 2^{-\alpha(1-c\rho)n}$ .

If  $\eta < 1$  then, applying Lemma 2.1 with  $\delta = 1$ , we obtain

$$\begin{aligned}
\Delta L_{n,l,r}^-(T_-^{n,l,r}(s)) &\leq C(t-s)^{\frac{1}{\beta+1}-\gamma} |y-x|^{\eta-\eta_c-2\rho} \frac{4n^q 2^{-n}}{(t-s)^{2/\alpha}} \\
&\leq Cn^q 2^{-n} (t-s)^{\frac{1}{\beta+1}-\gamma-2/\alpha} (t-s)^{(1-\nu)(\eta-\eta_c-2\rho)/\alpha} \\
&= Cn^q 2^{-n} \left( (t-s)^{1/\alpha} \right)^{\frac{\alpha}{\beta+1}-\alpha\gamma-2+(1-\nu)(\eta-\eta_c-2\rho)} \\
&= Cn^q 2^{-n} \left( (t-s)^{1/\alpha} \right)^{\eta-1-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)} \\
&\leq Cn^q 2^{-n} \left( 2^{-n(1-c\rho)} \right)^{\eta-1-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.
\end{aligned}$$

If  $\eta > 1$ , then we can apply Corollary 2.3 with  $\delta = 2$ , which gives

$$\begin{aligned}
\Delta L_{n,l,r}^-(T_-^{n,l,r}(s)) &\leq C(t-s)^{\frac{1}{\beta+1}-\gamma} |y-x|^{\eta-\eta_c-2\rho} \frac{16n^{2q} 2^{-2n}}{(t-s)^{3/\alpha}} \\
&\leq Cn^{2q} 2^{-2n} (t-s)^{\frac{1}{\beta+1}-\gamma-3/\alpha} (t-s)^{(1-\nu)(\eta-\eta_c-2\rho)/\alpha} \\
&= Cn^{2q} 2^{-2n} \left( (t-s)^{1/\alpha} \right)^{\frac{\alpha}{\beta+1}-\alpha\gamma-3+(1-\nu)(\eta-\eta_c-2\rho)} \\
&= Cn^{2q} 2^{-2n} \left( (t-s)^{1/\alpha} \right)^{\eta-2-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)} \\
&\leq Cn^{2q} 2^{-2n} \left( 2^{-n(1-c\rho)} \right)^{\eta-2-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.
\end{aligned}$$

Hence for  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ , and with  $c, \rho, \nu$  as in (5.30), we immediately get (5.38) with an appropriate constant  $C_{(5.38)}$ .

Now we consider the complimentary subset of  $S_{n,x}^2$ , where

$$|y-x| \leq (t-s)^{(1-\nu)/\alpha} \quad \text{and} \quad s \geq t - 2^{-\alpha(1+c\rho)n}.$$

It follows from the definition of  $\tilde{p}_{t-s}^{\alpha,\eta}$  that, for  $\eta > 1$ ,

$$|\tilde{p}_{t-s}^{\alpha,\eta}(l2^{-Qn} - y, r2^{-Qn} - y)| \leq \frac{2p_1^\alpha(0)}{(t-s)^{1/\alpha}} + (r-l)2^{-Qn} \sup_z \left| \frac{\partial}{\partial z} p_1^\alpha(z) \right|. \quad (5.45)$$

With this and recalling that  $(r-l)2^{-Qn} \leq 4n^q 2^{-n}$ , we obtain

$$\begin{aligned}
\Delta L_{n,l,r}^-(T_-^{n,l,r}(s)) &\leq C(t-s)^{\frac{1}{\beta+1}-\gamma} |y-x|^{\eta-\eta_c-2\rho} \left( (t-s)^{-1/\alpha} + n^q 2^{-n} (t-s)^{-2/\alpha} \right) \\
&\leq Cn^q 2^{-n} (t-s)^{\frac{1}{\beta+1}-\gamma-2/\alpha} |y-x|^{\eta-\eta_c-2\rho} \\
&\leq Cn^q 2^{-n} \left( (t-s)^{1/\alpha} \right)^{\eta_c-1-\alpha\gamma+(1-\nu)(\eta-\eta_c-2\rho)} \\
&\leq Cn^q 2^{-n} \left( 2^{-(1+c\rho)n} \right)^{\eta-1-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.
\end{aligned}$$

Again, with  $c, \rho, \nu$  as in (5.30), we immediately get (5.38) with an appropriate constant  $C_{(5.38)}$ . If  $\eta < 1$  then, instead of (5.45), we have a simpler inequality

$$|\tilde{p}_{t-s}^{\alpha,\eta}(l2^{-Qn} - y, r2^{-Qn} - y)| \leq \frac{2p_1^\alpha(0)}{(t-s)^{1/\alpha}}.$$



Thus,

$$\Delta L_{n,l,r}^-(T_{-}^{n,l,r}(s)) \leq C \left( 2^{-(1+c\rho)n} \right)^{\eta-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.$$

Consequently, (5.38) holds also for  $\eta < 1$ .

(iii) Let  $(s, y) \in S_{n,x}^3 \cap S^4$ .

Recall that on this set,  $\Delta X_s(y) \leq (t-s)^{(\eta+1+3\rho)/\alpha}$ . Then applying Corollary 2.3 (if  $\eta > 1$ ) or Lemma 2.1 (if  $\eta < 1$ ) with  $\delta = \eta + 3\rho$ , and by using that  $c, \rho, \nu$  are as in (5.30), one can easily get (5.38) in this case as well.  $\square$

Recall that  $\mathbf{L}_{n,l,r}^- = L_{n,l,r}^-(T_{-}^{n,l,r}(t))$ . In the next lemma, we will deal with regions where  $\left\{ \mathbf{L}_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^- \right\}_{n \geq 1}$  may take "big" values infinitely often.

**Lemma 5.13.** *For any  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ , we have*

$$\mathbf{P} \left( \left\{ x \in (0,1) \setminus S_{\eta-2\rho} : \left\{ \mathbf{L}_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^- \geq 2^{-\eta n-1} \text{ and } 1\{A_{k_n(x)}^{(n)}\} = 1 \right\} \text{ i.o.} \right\} \subset \tilde{G}_\eta | A^\varepsilon \right) = 1.$$

*Proof.* First we will show that on  $\omega \in A^\varepsilon$

$$\begin{aligned} \{x \in (0,1) \setminus S_{\eta-2\rho} : \Delta L_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^- \geq C_{(5.38)} 2^{-(\eta+2\rho)n} \\ \text{and } 1\{A_{k_n(x)}^{(n)}\} = 1 \text{ i.o.}\} \subset \tilde{G}_\eta. \end{aligned} \quad (5.46)$$

On  $\omega \in A^\varepsilon$ , take  $n \geq N_{(5.12)}$ , and fix some  $x \in (0,1) \setminus S_{\eta-2\rho}$  satisfying  $1\{A_{k_n(x)}^{(n)}\} = 1$ . First of all, by definition of  $A_{k_n(x)}^{(n)}$ , if  $1\{A_{k_n(x)}^{(n)}\} = 1$  then there exists a jump of  $M$  of the form  $\tilde{r}\delta_{(\tilde{s},\tilde{y})}$  with  $\tilde{r}, \tilde{s}, \tilde{y}$  as in  $G_{k_n(x)}^{(n)}$  (see (5.31)). Moreover, again by the definition of  $A_{k_n(x)}^{(n)}$ , the spatial position of the jump,  $\tilde{y}$ , is in  $I_{k_n(x)-2n^q-2}^{(n)}$ . Hence it is easy to see that this jump does not contribute to the jumps of  $L_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^-$  that is,  $\Delta L_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^-(T_{-}^{n,\tilde{l}_n(x),\tilde{r}_n(x)}(\tilde{s})) = 0$ . Thus, we have to show that, if  $\Delta L_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^- \geq C_{(5.38)} 2^{-(\eta+2\rho)n}$ , and  $1\{A_{k_n(x)}^{(n)}\} = 1$ , then there exists at least one another big jump of  $M$  with properties described in  $G_k^{(n)}$ .

By Lemma 5.12, we get that if there exists  $s$  such that

$$\Delta L_{n,\tilde{l}_n(x),\tilde{r}_n(x)}^-(T_{-}^{n,\tilde{l}_n(x),\tilde{r}_n(x)}(s)) \geq C_{(5.38)} 2^{-(\eta+2\rho)n},$$

then the corresponding jump of  $M$ , of the form  $r\delta_{(s,y)}$ , has to satisfy

$$|y-x| \leq (t-s)^{(1-\nu)/\alpha}, \quad s \in [t-2^{-\alpha(1-c\rho)n}, t-2^{-\alpha(1+c\rho)n}], \quad r \geq (t-s)^{(\eta+1+3\rho)/\alpha}.$$

This yields that on  $A^\varepsilon$  (5.46) holds.

Second, it follows from Lemma 2.5 and (5.29), that

$$\mathbf{P} \left( \mathbf{L}_{n,l,r}^- \geq 2^{-\eta n-1}; \Delta L_{n,l,r}^- \leq C_{(5.38)} 2^{-(\eta+2\rho)n} \right) \leq \exp\{-c2^{2\rho n}\}$$

for all  $l, r$  satisfying  $(r-l)2^{-Qn} \leq C2^{-n}$ .

Applying now Borel-Cantelli we conclude that, with probability one,

$$\bigcup_{0 \leq l < r \leq 2^{Qn-1}, (r-l)2^{-Qn} \leq C2^{-n}} \left\{ \mathbf{L}_{n,l,r}^- \geq 2^{-\eta n-1}; \Delta L_{n,l,r}^- \leq C_{(5.38)} 2^{-(\eta+2\rho)n} \right\}$$

occurs only finite number of times. This completes the proof of the lemma.  $\square$

*Proof of Proposition 5.1.* It follows from Lemmas 5.8 and 5.13, that for every  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ ,  $x \in (\tilde{V}_\eta \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta$ , for  $\mathbf{P}$ -a.s.  $\omega$  on  $A^\varepsilon$ , there exists a (random) sequence  $\{n_j\}_{j \geq 1}$  such that

$$\mathbf{L}_{n_j, \tilde{l}^{n_j}(x), \tilde{r}^{n_j}(x)}^+ \geq n_j^m 2^{-\eta n_j}, \quad \mathbf{L}_{n_j, \tilde{l}^{n_j}(x), \tilde{r}^{n_j}(x)}^- \leq 2^{-(\eta n_j - 1)},$$

for all  $n_j$  sufficiently large. Note that the first inequality holds for every  $\eta \in (\eta_c, \bar{\eta}_c)$ . This implies that, on the event  $A^\varepsilon$ , we have

$$\liminf_{j \rightarrow \infty} 2^{(\eta+\delta)n_j} \left| \tilde{Z}_t^{2,\eta} \left( \frac{\tilde{l}^{n_j}(x)}{2^{Qn_j}}, \frac{\tilde{r}^{n_j}(x)}{2^{Qn_j}} \right) \right| = \infty. \quad (5.47)$$

for any  $\delta > 0$ . Recall that,  $X_t(\cdot)$  and  $Z_t^2(\cdot)$  are Hölder continuous with any exponent less than  $\eta_c$  at every point of  $(0, 1)$ . Therefore, by choosing  $Q > 4\frac{\eta}{\eta_c}$ , we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \sup_{x \in (0,1)} 2^{(\eta+\delta)n_j} |X_t(x) - X_t(\tilde{r}^{n_j}(x) 2^{-Qn_j})| &= \lim_{j \rightarrow \infty} C(\omega) 2^{-\frac{1}{2}Q\eta_c n_j} 2^{(\eta+\delta)n_j} \\ &= 0 \quad \mathbf{P} - \text{a.s. on } A^\varepsilon. \end{aligned} \quad (5.48)$$

If  $\eta < 1$ , then  $\tilde{Z}_t^{2,\eta}(x_1, x_2) = Z_t^2(x_1) - Z_t^2(x_2)$ . Therefore, combining (5.47) and (5.48), we conclude that

$$H_{Z^2}(x) \leq \eta \quad \text{for all } x \in (\tilde{V}_\eta \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta \quad \mathbf{P} - \text{a.s. on } A^\varepsilon. \quad (5.49)$$

Assume now that  $\eta > 1$ . In this case we infer from (4.28) that

$$\limsup_{j \rightarrow \infty} \sup_{x \in (0,1) \setminus S_{\eta-2\rho}} 2^{Q(\eta-1-2\rho-2\alpha\gamma)n_j} |V'(\tilde{r}^{n_j} 2^{-Qn_j}) - V'(x)| = 0 \quad \mathbf{P} - \text{a.s. on } A^\varepsilon. \quad (5.50)$$

Combining (5.47), (5.48) and (5.50), and choosing  $Q > 4\eta/(\eta-1)$ , we get

$$\liminf_{j \rightarrow \infty} 2^{(\eta+\delta)n_j} \left| \tilde{Z}_t^{2,\eta} \left( 2^{-Qn_j} \tilde{l}^{n_j}(x), x \right) \right| = \infty, \quad \text{on } A^\varepsilon, \quad \mathbf{P} - \text{a.s.},$$

This implies that (5.49) holds for all  $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ .

We know, by Lemma 4.2, that

$$H_{Z^2}(x) \geq \eta - \alpha\gamma - 2\rho \quad \text{for all } x \in (0, 1) \setminus S_{\eta-2\rho}, \quad \mathbf{P} - \text{a.s.},$$

This and (5.49) imply that on  $A^\varepsilon$ ,  $\mathbf{P} - \text{a.s.}$ ,

$$\eta - \alpha\gamma - 2\rho \leq H_{Z^2}(x) \leq \eta \quad \text{for all } x \in (\tilde{V}_\eta \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta, \quad \forall \eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}, \quad (5.51)$$

It follows easily from Lemma 4.3, Corollary 5.6 and Lemma 5.9 that on  $A^\varepsilon$

$$\dim\left((\tilde{V}_\eta \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta\right) \geq (\beta+1)(\eta - \eta_c), \quad \mathbf{P} - \text{a.s.}$$

Thus, by (5.51),

$$\dim\{x : H_{Z^2}(x) \leq \eta\} \geq (\beta+1)(\eta - \eta_c), \quad \text{on } A^\varepsilon, \quad \mathbf{P} - \text{a.s.}$$

It is clear that

$$\begin{aligned} \{x : H_{Z^2}(x) = \eta\} \cup \bigcup_{n=n_0}^{\infty} \{x : H_{Z^2}(x) \in (\eta - n^{-1}, \eta - (n+1)^{-1}]\} \\ = \{x : \eta - n_0^{-1} \leq H_{Z^2}(x) \leq \eta\}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \mathcal{H}_\eta(\{x : \eta - n_0^{-1} \leq H_{Z^2}(x) \leq \eta\}) \\ &= \mathcal{H}_\eta(\{x : H_{Z^2}(x) = \eta\}) + \sum_{n=n_0}^{\infty} \mathcal{H}_\eta(\{x : H_{Z^2}(x) \in (\eta - n^{-1}, \eta - (n+1)^{-1}]\}). \end{aligned}$$

Since the dimensions of  $S_{\eta-2\rho}$  and  $\tilde{G}_\eta$  are smaller than  $\eta$ , the measure  $\mathcal{H}_\eta$  of that sets equals zero. Applying Corollary 5.6, we then conclude that on  $A^\varepsilon$

$$\mathcal{H}_\eta((\tilde{V}_\eta \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta) > 0, \mathbf{P} - \text{a.s.}$$

And in view of (5.51),  $\mathcal{H}_\eta(\{x : \eta - n_0^{-1} \leq H_{Z^2}(x) \leq \eta\}) > 0$ . Furthermore, it follows from Proposition 4.1, that dimension of the set  $\{x : H_{Z^2}(x) \in (\eta - n^{-1}, \eta - (n+1)^{-1}]\}$  is bounded from above by  $(\beta + 1)(\eta - (n+1)^{-1} - \eta_c)$ . Hence, the definition of  $\mathcal{H}_\eta$  immediately yields

$$\mathcal{H}_\eta(\{x : H_{Z^2}(x) \in (\eta - n^{-1}, \eta - (n+1)^{-1}]\}) = 0, \text{ on } A^\varepsilon, \mathbf{P} - \text{a.s.},$$

for all  $n \geq n_0$ . As a result we have

$$\mathcal{H}_\eta(\{x : H_{Z^2}(x) = \eta\}) > 0 \quad \mathbf{P} - \text{a.s. on } A^\varepsilon. \quad (5.52)$$

Since  $\varepsilon > 0$  was arbitrary, this implies that (5.52) is satisfied on the whole probability space  $\mathbf{P}$ -a.s. From this we get that

$$\dim\{x : H_{Z^2}(x) = \eta\} \geq (\beta + 1)(\eta - \eta_c), \quad \mathbf{P} - \text{a.s.}$$

□

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